

SOLUTIONS MANUAL

**CONTINUOUS
AND
DISCRETE SIGNALS
AND
SYSTEMS**

S E C O N D E D I T I O N

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CONTENTS

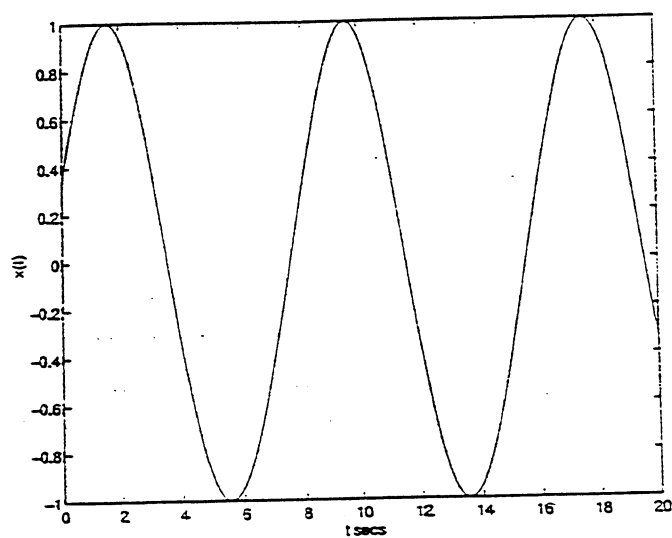
Chapter 1	1
Chapter 2	22
Chapter 3	50
Chapter 4	87
Chapter 5	111
Chapter 6	138
Chapter 7	171
Chapter 8	192
Chapter 9	219
Chapter 10	231

Chapter 1

1.1

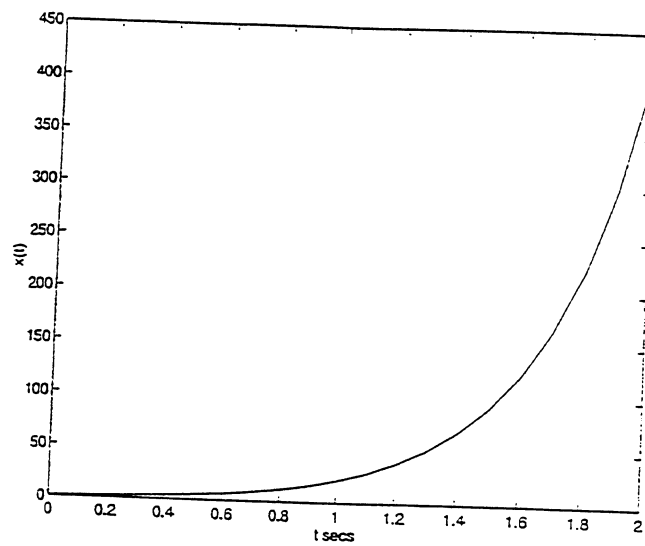
Signal	Period
$\cos(\pi t)$	2
$\sin(2\pi t)$	1
$\cos(3\pi t)$	$\frac{1}{3}$
$\sin(4\pi t)$	$\frac{1}{2}$
$\cos(\frac{\pi}{2} t)$	4
$\sin(\frac{\pi}{3} t)$	6
$\cos(\frac{5\pi}{2} t)$	$\frac{4}{5}$
$\sin(\frac{4\pi}{3} t)$	$\frac{3}{2}$
$\cos(\frac{\pi}{4} t)$	8
$\sin(\frac{2\pi}{3} t)$	$\frac{1}{3}$
$\cos(\frac{3\pi}{5} t)$	$\frac{10}{3}$

1.2

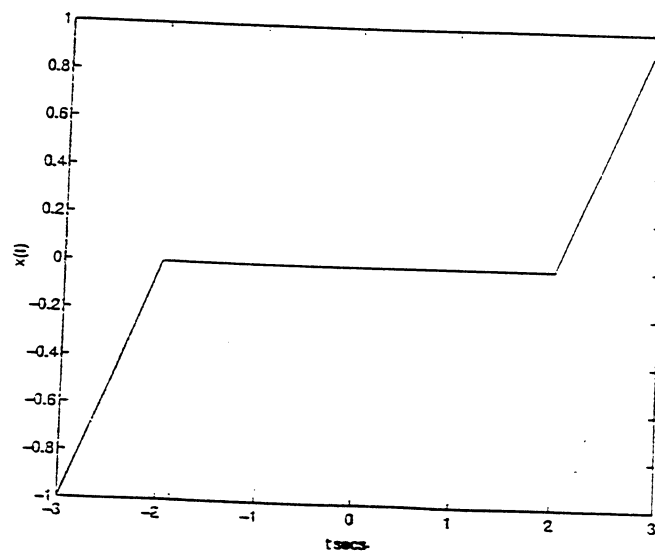


(a) $x(t) = \sin(\frac{\pi}{4} t + 20^\circ)$

1.2

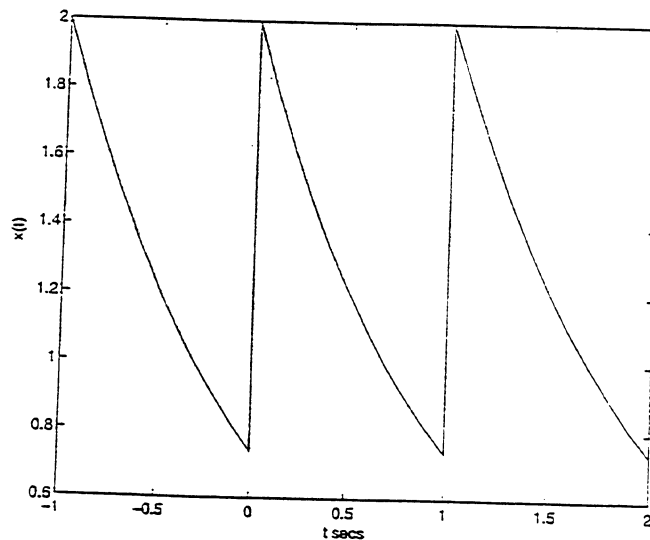


(b) $x(t) = t + e^{3t}$



$$(c) x(t) = \begin{cases} t+2 & t \leq -2 \\ 0 & -2 \leq t \leq 2 \\ t-2 & 2 \leq t \end{cases}$$

1.2



(d) $x(t) = 2\exp[-t]$, $0 \leq t < 1$, and $x(t+1) = x(t)$

1.3 $x(t+T) = x(t)$ for every t .

if we substitute $t = t+T$

$$x(t+2T) = x(t+T) = x(t)$$

i.e $x(t)$ is periodic with period $2T$

if we substitute $t = t+2T$

$$x(t+3T) = x(t+2T) = x(t)$$

i.e $x(t)$ is periodic with period $3T$

In general $x(t)$ is periodic with period nT

1.4 $x_3(t) = a x_1(t) + b x_2(t)$

$$x_3(t+T) = a x_1(t+T) + b x_2(t+T)$$

$$= a x_1(t) + b x_2(t)$$

$$= x_3(t)$$

1.5 (a) $\sin \frac{2\pi}{3}t$ is periodic with period 3 while $\sin \frac{16\pi}{3}t$ is periodic with period $\frac{3}{8}$. Then $x(t)$ is periodic with period 3.

1.5 (b) $\exp[j\frac{2\pi}{6}t]$ is periodic with period 6 and

$\exp[-j\pi t]$ is periodic with period 2.

Then $x(t)$ is periodic with period 6.

(c) $\exp[j\frac{2\pi}{5}t]$ is periodic 5 while

$\exp[j3t]$ is periodic with period $\frac{2\pi}{3}$

Since the ratio of these two periods, $\frac{5}{\frac{2\pi}{3}} = \frac{15}{2\pi}$

is not a rational number, $x(t)$ is not periodic.

1.6 (a) $T_1 = 6, T_2 = \frac{3}{4} \frac{T_1}{T_2} = 8$. Periodic, with period $T = 6$.

(b) $T_1 = \frac{7}{6}, T_2 = \frac{5}{6} \frac{T_1}{T_2} = \frac{7}{5}$. Periodic, with period $T = 35$.

(d) $\exp[\frac{5}{6}t]$ is not periodic, so that $x(t)$ is not periodic.

(e) $T_1 = \frac{16}{3}, T_2 = \frac{8\pi}{3} \frac{T_1}{T_2} = \frac{8\pi}{3}$ is not a rational number. $x(t)$ is not periodic.

1.7 $\exp[j\omega t] = \cos \omega t + j \sin \omega t$ (Euler's formula)

Since $\cos \omega t$ and $\sin \omega t$ are periodic with period $2\pi/\omega$, then their linear combination is also periodic with period $2\pi/\omega$.

1.8 Since $x(t)$ is periodic with period T , then

$$\begin{aligned} x(at) &= x(at+T) \\ &= x(a(t+T/a)) \end{aligned}$$

For $x(at)$ to be periodic with period T_1 , one needs $T_1 = T/a$

Similarly,

$$\begin{aligned} x(t/b) &= x(t/b + T) \\ &= x\left(\frac{1}{b}(t+bT)\right) \end{aligned}$$

For $x(t/b)$ to be periodic with period T_2 , one needs

$$T_2 = bT$$

For $\sin t$, $T = 2\pi$

For $\sin 2t$, $T_1 = \pi = T/2$

For $\sin t/2$, $T_2 = 4\pi = 2T$

1.9 (a)

$$\begin{aligned} P &= \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T t^2 \sin^2\left(\frac{\pi}{3}t\right) dt \\ &= \lim_{T \rightarrow \infty} \frac{1}{4T} \int_{-T}^T t^2 (1 - \cos(\frac{2\pi}{3}t)) dt \\ &= \lim_{T \rightarrow \infty} \frac{1}{4T} \left[\frac{2T^3}{3} - \int_{-T}^T \cos(\frac{2\pi}{3}t) dt \right] \rightarrow \infty \end{aligned}$$

so that $x(t)$ is neither an energy nor a power signal.

(b)

$$\begin{aligned} E &= \lim_{T \rightarrow \infty} \int_{-T}^T \exp[-4|t|] \sin^2(\pi t) dt \\ &= \lim_{T \rightarrow \infty} \int_0^T \exp[-4t] [1 - \cos(2\pi t)] dt = \frac{\pi^2}{\pi^2 + 4} \end{aligned}$$

so that $x(t)$ is an energy signal with $P = 0$.

$$(c) \quad P = \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T \exp(8|t|) dt = \lim_{T \rightarrow \infty} \frac{e^{8T} - 1}{T} \rightarrow \infty$$

so that $x(t)$ is neither an energy nor a power signal.

(d) $x(t)$ is periodic with period $T = 12/5$ s, so that it is a power signal with

$$P = \frac{1}{T} \int_0^T |x(t)|^2 dt = \frac{1}{T} \int_0^T \exp[j \frac{5\pi}{6} t]^2 dt = 1$$

(e) $x(t) = \frac{1}{2} [\sin(\frac{25\pi}{24} t) - \sin(\frac{7\pi}{24} t)]$ and is periodic with $T = 48$ secs. Thus

$$\begin{aligned} P &= \frac{1}{4T} \int_0^T [\sin(\frac{25\pi}{24} t) - \sin(\frac{7\pi}{24} t)]^2 dt \\ &= \frac{1}{4T} \int_0^T \sin^2(\frac{25\pi}{24} t) dt + \frac{1}{4T} \int_0^T \sin^2(\frac{7\pi}{24} t) dt - \frac{1}{2T} \int_0^T \sin(\frac{25\pi}{24} t) \sin(\frac{7\pi}{24} t) dt \\ &= \frac{1}{8} + \frac{1}{8} = \frac{1}{4} \end{aligned}$$

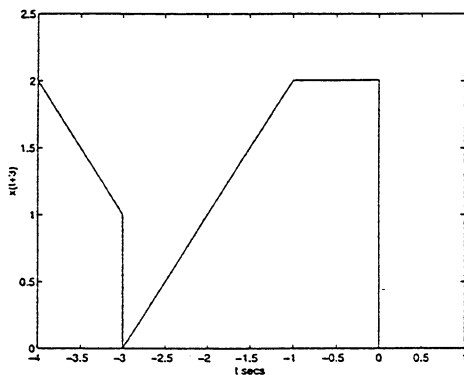
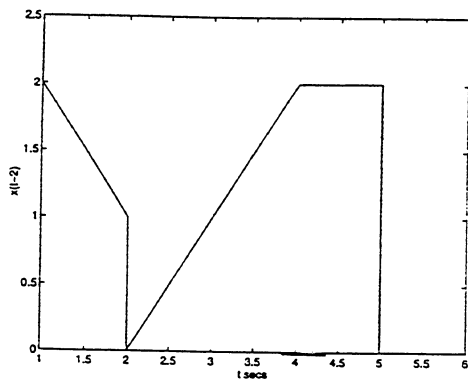
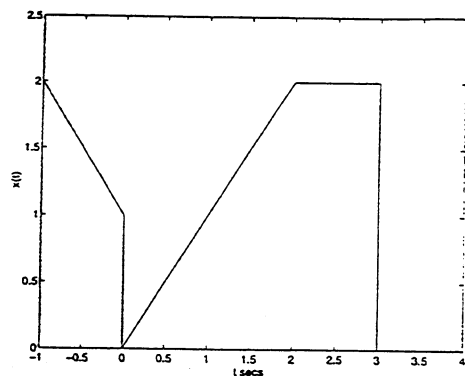
$$\begin{aligned} (f) \quad P &= \lim_{T \rightarrow \infty} \frac{1}{2T} \left[\int_{-T}^0 1 dt + \int_0^T e^{-6t} dt \right] \\ &= \lim_{T \rightarrow \infty} \frac{1}{2T} \left[T + \frac{1 - e^{-6T}}{6} \right] = \frac{1}{2} \end{aligned}$$

so that $x(t)$ is a power signal.

$$1.10 \quad \left| \int_0^T x(t) dt \right|^2 \leq \int_0^T |x(t)|^2 dt = PT$$

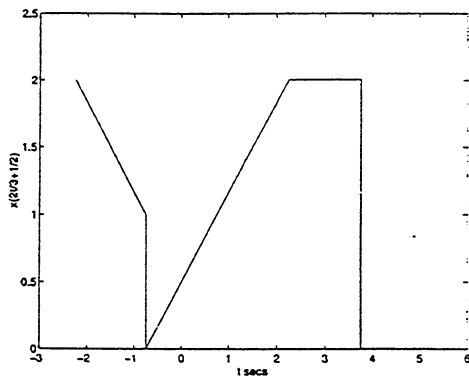
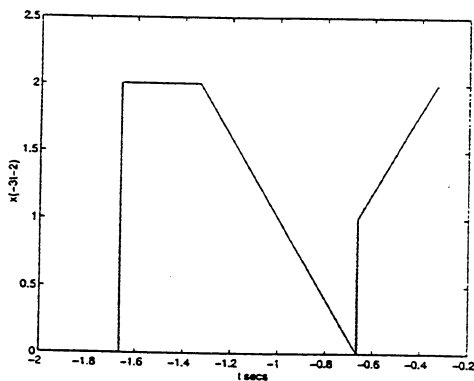
$$\left| \int_0^T x(t) dt \right| \leq \sqrt{PT}$$

1.11



$$x(t-2) = \begin{cases} -t+3 & 1 \leq t < 2 \\ t-2 & 2 \leq t < 4 \\ 2 & 4 \leq t < 5 \\ 0 & \text{otherwise} \end{cases}$$

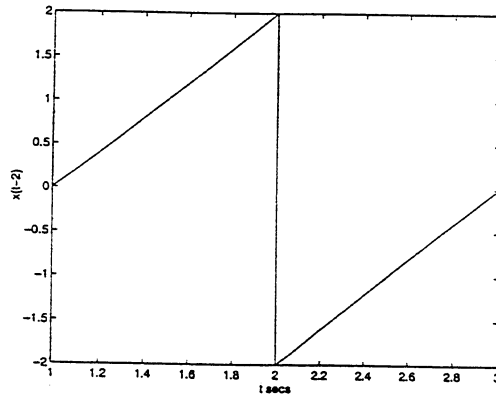
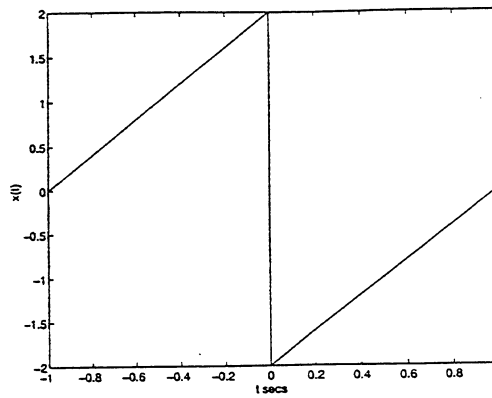
$$x(t+3) = \begin{cases} -t-3 & -4 \leq t < -3 \\ t+3 & -3 \leq t < -1 \\ 2 & -1 \leq t < 0 \\ 0 & \text{otherwise} \end{cases}$$



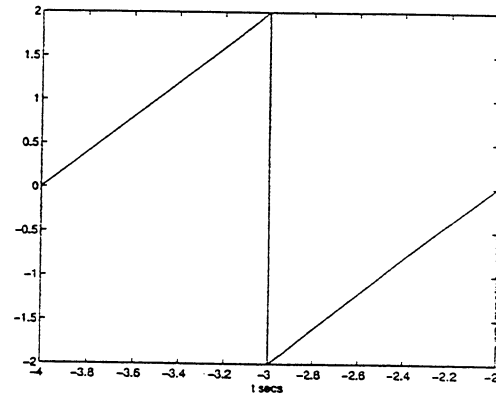
$$x(-3t-2) = \begin{cases} 3t+5 & -\frac{2}{3} < t \leq -\frac{1}{3} \\ -3t-2 & -\frac{4}{3} < t \leq -\frac{2}{3} \\ 2 & -\frac{5}{3} < t \leq -\frac{4}{3} \\ 0 & \text{otherwise} \end{cases}$$

$$x\left(\frac{2}{3}t + \frac{1}{2}\right) = \begin{cases} \frac{2}{3}t + \frac{1}{2} & -\frac{9}{4} \leq t < -\frac{3}{4} \\ \frac{2}{3}t + \frac{1}{2} & -\frac{3}{4} \leq t < \frac{9}{4} \\ 2 & \frac{9}{4} \leq t < \frac{15}{4} \\ 0 & \text{otherwise} \end{cases}$$

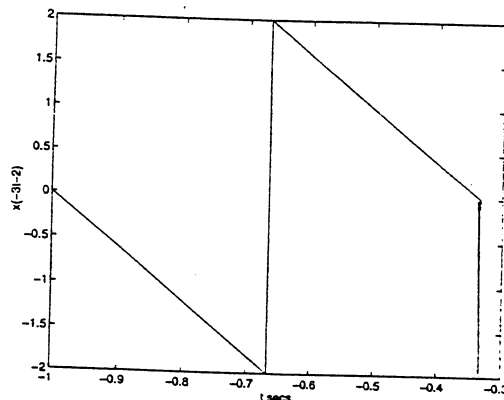
1.12



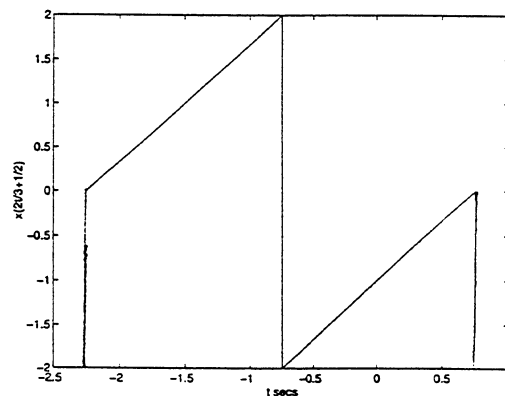
$$x(t-2) = \begin{cases} 2t-2 & 1 \leq t < 2 \\ 2t-6 & 2 \leq t < 3 \\ 0 & \text{otherwise} \end{cases}$$



$$x(t+3) = \begin{cases} 2t+8 & -4 \leq t < -3 \\ 2t+4 & -3 \leq t < -2 \\ 0 & \text{otherwise} \end{cases}$$

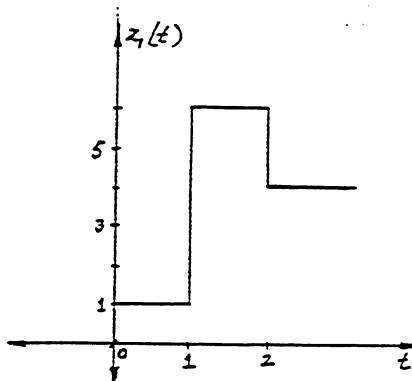


$$x(-3t-2) = \begin{cases} -6t-6 & -1 < t \leq -\frac{2}{3} \\ -6t-2 & -\frac{2}{3} < t \leq -\frac{1}{3} \\ 0 & \text{otherwise} \end{cases}$$

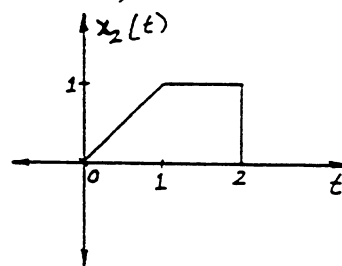


$$x\left(\frac{2}{3}t + \frac{1}{2}\right) = \begin{cases} \frac{4}{3}t+3 & -\frac{9}{4} \leq t < -\frac{3}{4} \\ \frac{4}{3}t-1 & -\frac{3}{4} \leq t < \frac{3}{4} \\ 2 & \text{otherwise} \end{cases}$$

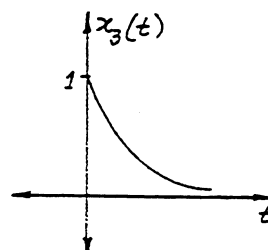
$$1.13(a) \quad x_1(t) = u(t) + 5u(t-1) - 2u(t-2)$$



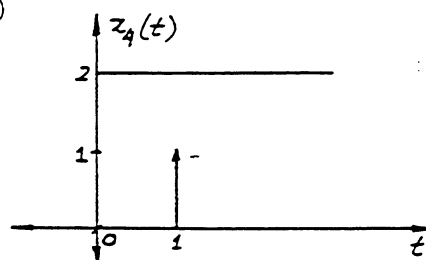
$$1.13(b) \quad x_2(t) = r(t) - r(t-1) - u(t-2)$$



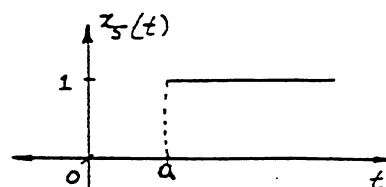
$$(c) \quad x_3(t) = e^{-t} u(t)$$



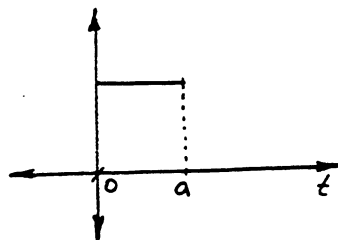
$$(d) \quad x_4(t) = 2u(t) + \delta(t-1)$$



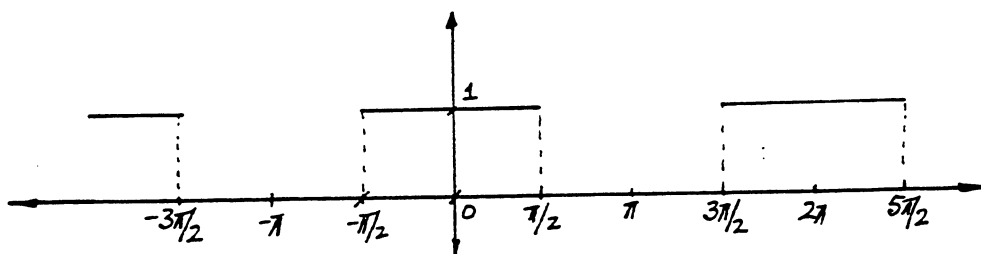
$$(e) \quad x_5(t) = u(t) u(t-a), \quad a > 0$$



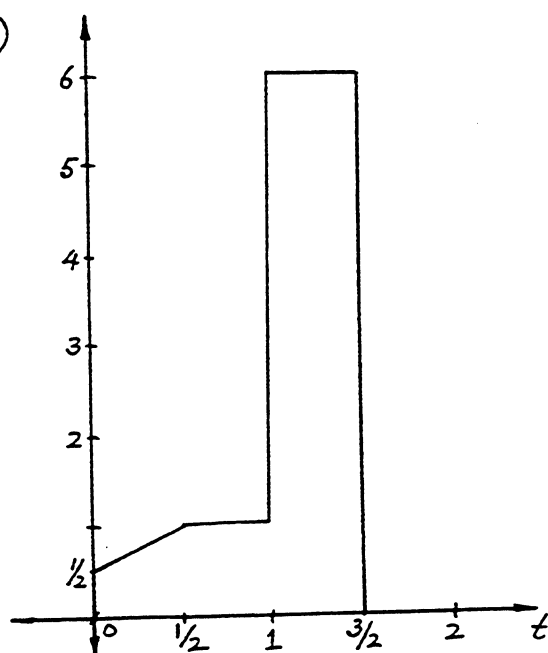
(f) $x_c(t) = u(t)u(a-t)$, $a > 0$



(g) $x_7(t) = u(\cos t) = \begin{cases} 1 & \cos t \geq 0 \\ 0 & \text{o.w.} \end{cases}$

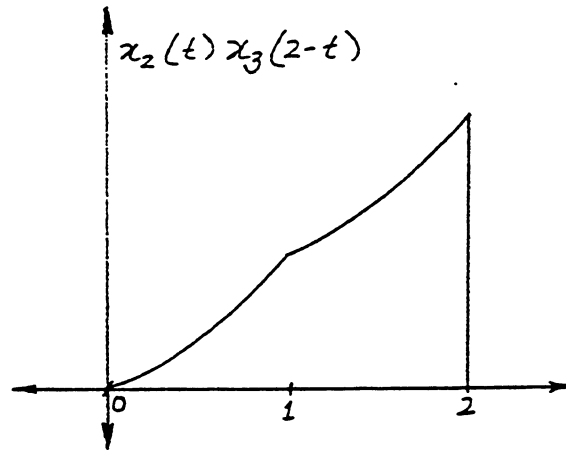


(h) $x_1(t) x_2(t + \frac{1}{2})$



$$(i) \quad x_1\left(-\frac{t}{3} + \frac{1}{2}\right) x_3(t-2) = 0 \quad \text{for all } t$$

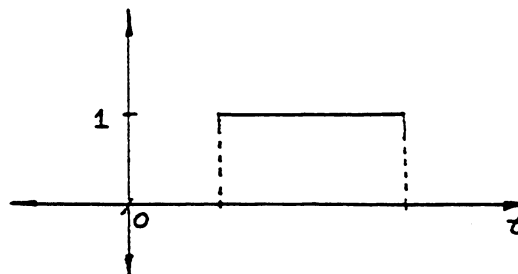
(j)



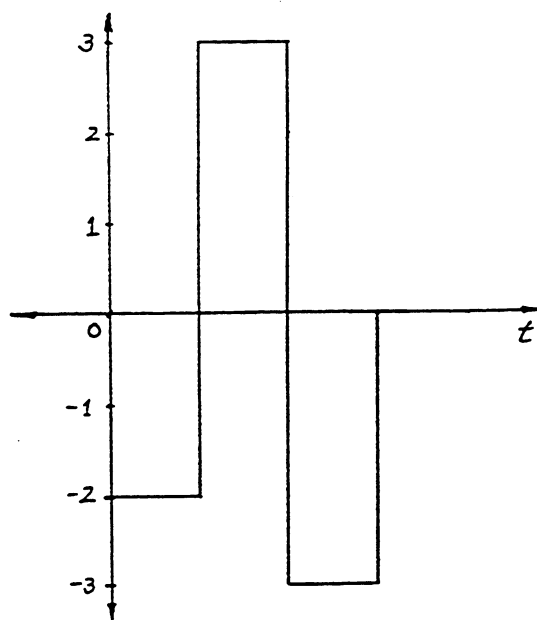
$$\begin{aligned} 1.14 (a) \quad x_c(t) &= \frac{1}{2} [x(-t) + x(-t)] \\ &= x_c(t) \end{aligned}$$

$$\begin{aligned} (b) \quad x_o(-t) &= \frac{1}{2} [x(-t) - x(t)] \\ &= -\frac{1}{2} [x(t) - x(-t)] \\ &= -x_o(t) \end{aligned}$$

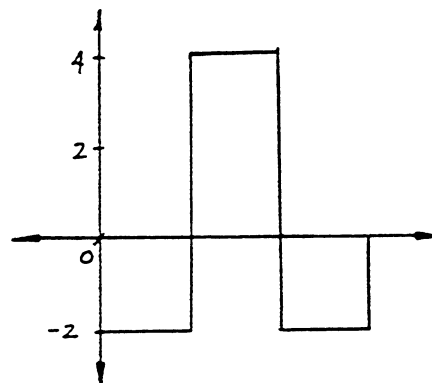
$$1.15 (a) (i) \quad L+R$$



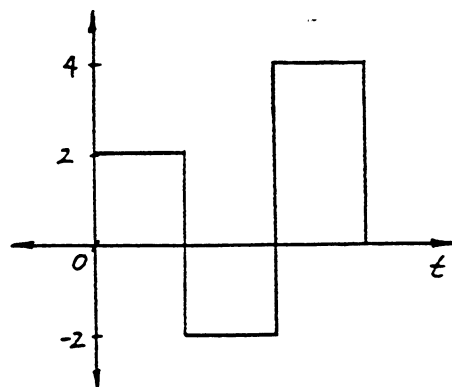
(ii) $L - R$



(b) $(L + R) + (L - R) = 2L$



(c) $(L + R) + (-(L - R)) = (L + R) + (R - L) = 2R$



$$1.16 \quad x_1(t) = u(t) - u(t-a)$$

$$x_2(t) = r(t) - r(t-a) - a u(t-b)$$

$$x_3(t) = \frac{a}{b-a} [r(t+b) - r(t+a) - r(t-a) + r(t-b)]$$

$$x_4(t) = (r(t) - r(-t)) (u(t+a) - u(t-a))$$

$$x_5(t) = \frac{1}{c} r(t) - \frac{a}{c(a-c)} r(t-c) + \frac{1}{c} r(t-2a+c)$$

$$x_6(t) = \frac{1}{b-a} [r(t+b) - r(t+a)] + u(t) - \frac{2}{b-a} [r(t-a) - r(t-b)]$$

$$1.17 (a) \quad x_1(t) = A e^{-t/T} u(t)$$

$$x_1(0) = A \quad x_1(t') = \frac{A}{e} = A e^{-t'/T}$$

$$\therefore t' = T$$

The duration of $x_1(t) = T$

$$(b) \quad x_2(t) = x_1(3t) = A e^{-3t/T} u(3t) = A e^{-3t/T} u(t)$$

$$x_2(t') = \frac{A}{e} = A e^{-3t'/T} \Rightarrow t' = \frac{T}{3}$$

The duration of $x_2(t) = \frac{T}{3}$

$$(c) \quad x_3(t) = x_1(t/2) = A e^{-t/2T} u(t/2) = A e^{-t/2T} u(t)$$

$$x_3(0) = A$$

$$x_3(t') = \frac{A}{e} = A e^{-t'/2T} \Rightarrow t' = 2T$$

The duration of $x_3(t) = 2T$

$$(d) X_4(t) = 2X_1(t) = 2Ae^{-t/T} u(t)$$

$$X_4(0) = 2A$$

$$X_4(t') = \frac{2A}{e} = 2Ae^{-t'/T} \Rightarrow t' = T$$

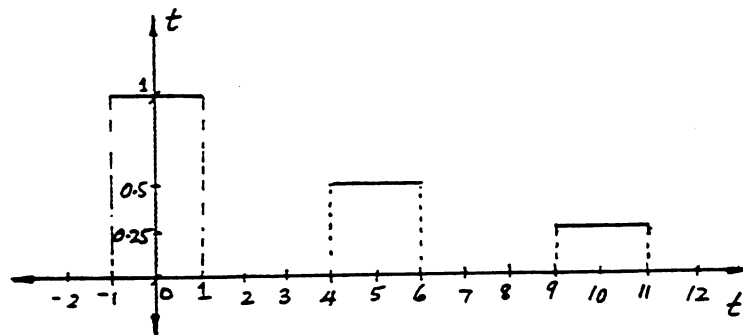
The duration of $X_4(t) = T$

$$1.18 \quad X(t) = \text{rect}(t/2)$$

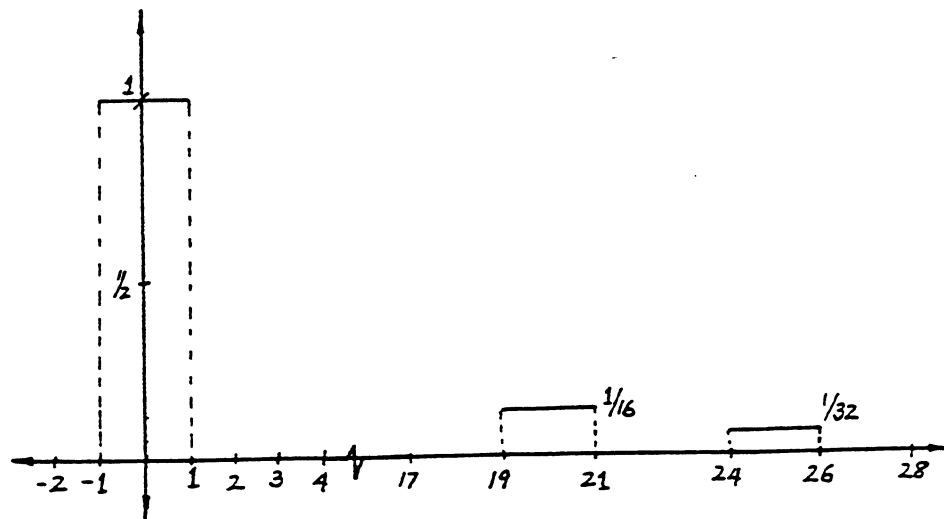
$$y(t) = X(t) + 0.5X(t-T/2) + 0.25X(t-T) \quad T \gg 2$$

$$(a) \quad T = 10$$

$$y(t) = X(t) + 0.5X(t-5) + 0.25X(t-10) \quad T \gg 2$$



$$(b) \quad T = 10, \quad z(t) = y(t) - \frac{1}{2}y(t-5) + \frac{1}{8}y(t-15)$$



$$1.19 (a) P_\epsilon(t) = \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon \pi \cosh t/\epsilon}$$

note that $P_\epsilon(t)$ is an even symmetric function with

$$P_\epsilon(0) = \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon \pi} \rightarrow \infty$$

for $t \neq 0$

$$P_\epsilon(t) = \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon \pi \left(\frac{e^{t/\epsilon} + e^{-t/\epsilon}}{2} \right)}$$

$$= \lim_{\epsilon \rightarrow 0} \frac{2}{\epsilon \pi (e^{t/\epsilon} + e^{-t/\epsilon})}$$

$$= \lim_{\epsilon \rightarrow 0} \frac{2/\epsilon}{\pi (e^{t/\epsilon} + e^{-t/\epsilon})}$$

$$= \lim_{\epsilon \rightarrow 0} \frac{-\frac{2}{\epsilon^2}}{\pi (e^{t/\epsilon} + e^{-t/\epsilon}) \left(-\frac{1}{\epsilon^2}\right)}$$

$$= \lim_{\epsilon \rightarrow 0} \frac{1}{\pi (e^{t/\epsilon} + e^{-t/\epsilon})} = 0$$

$$\begin{aligned} \int_{-\infty}^{\infty} P_\epsilon(t) dt &= \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon \pi} \int_{-\infty}^{\infty} \frac{1}{\cosh(t/\epsilon)} dt \\ &= \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon \pi} \int_{-\infty}^{\infty} \operatorname{sech}(t/\epsilon) dt \\ &= \lim_{\epsilon \rightarrow 0} \frac{1}{\pi} \int_{-\infty}^{\infty} \operatorname{sech}(t') dt' \quad t' = \frac{t}{\epsilon} \\ &= \lim_{\epsilon \rightarrow 0} \frac{1}{\pi} \tan^{-1}(\sinh(t')) \Big|_{-\infty}^{\infty} \\ &= \lim_{\epsilon \rightarrow 0} \frac{1}{\pi} \left(\frac{\pi}{2} - \left(-\frac{\pi}{2}\right) \right) = 1 \end{aligned}$$

$\therefore P_\epsilon(t)$ can be used as a mathematical model for the delta function

$$\begin{aligned}
& \int_{-\infty}^{\infty} P_3(t) dt \\
&= \lim_{\epsilon \rightarrow 0} \int_{-\infty}^{\infty} \frac{2\epsilon}{\epsilon^2 + \pi^2 t^2} dt = \lim_{\epsilon \rightarrow 0} \int_{-\infty}^{\infty} \frac{1}{\pi} \frac{1}{1 + \left(\frac{2\pi t}{\epsilon}\right)^2} d\left(\frac{2\pi t}{\epsilon}\right) \\
&= \lim_{\epsilon \rightarrow 0} \frac{1}{\pi} \tan^{-1}\left(\frac{2\pi t}{\epsilon}\right) \Big|_{-\infty}^{\infty} \\
&= \frac{1}{\pi} \left(\frac{\pi}{2} - \left(-\frac{\pi}{2}\right) \right) = 1
\end{aligned}$$

$\therefore P_3(t)$ can be modelled as a delta function

$$(d) \quad P_4(t) = \lim_{\epsilon \rightarrow 0} \frac{1}{\pi} \frac{\epsilon}{t^2 + \epsilon^2}$$

note that $P_4(t)$ is an even symmetric function with

$$P_4(0) = \lim_{\epsilon \rightarrow 0} \frac{\epsilon}{\pi \epsilon^2} = 0$$

for $t \neq 0$

$$P_4(t) = \lim_{\epsilon \rightarrow 0} \frac{\epsilon}{\pi (t^2 + \epsilon^2)} = 0$$

$$\begin{aligned}
\int_{-\infty}^{\infty} P_4(t) dt &= \lim_{\epsilon \rightarrow 0} \int_{-\infty}^{\infty} \frac{\epsilon}{\pi (t^2 + \epsilon^2)} dt \\
&= \frac{1}{\pi} \tan^{-1} \frac{t}{\epsilon} \Big|_{-\infty}^{\infty} = \frac{1}{\pi} \left(\frac{\pi}{2} + \frac{\pi}{2} \right) = 1
\end{aligned}$$

$\therefore P_4(t)$ can be used as a model for the delta function.

$$(e) P_5(t) = \lim_{\epsilon \rightarrow 0} \epsilon \exp[-\epsilon/|t|]$$

note that $P_5(t)$ is an even symmetric function with

$$P_5(0) = \lim_{\epsilon \rightarrow 0} \epsilon = 0$$

$P_5(t)$ cannot be used as a model for the delta function.

$$(f) P_6(t) = \lim_{\epsilon \rightarrow 0} \frac{1}{\pi} \frac{\sin \epsilon t}{t}$$

note that $P_6(t)$ is an even symmetric function with

$$P_6(0) = \lim_{\epsilon \rightarrow 0} \frac{\epsilon}{\pi} \frac{\sin \epsilon t}{\epsilon t} = 0$$

$P_6(t)$ cannot be used as a model for the delta function.

$$1.20 \quad (a) \int_{-\infty}^{\infty} \left(\frac{2}{3}t - \frac{3}{2}\right) \delta(t-1) dt = \frac{2}{3} - \frac{3}{2} = -\frac{5}{6}$$

$$(b) \int_{-\infty}^{\infty} (t-1) \delta\left(\frac{2}{3}t - \frac{3}{2}\right) dt = \int_{-\infty}^{\infty} (t-1) \delta\left(\frac{2}{3}t - \frac{3}{2}\right) dt = \int_{-\infty}^{\infty} (t-1) \frac{3}{2} \delta(t-1) dt = 0$$

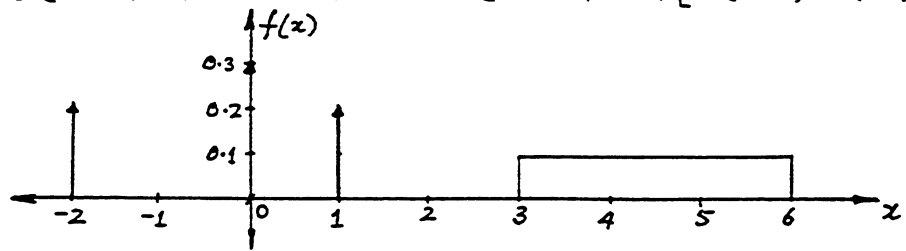
$$(c) \int_{-3}^{-2} [\exp(-t+1) + \sin\left(\frac{2\pi}{3}t\right)] \delta\left(t - \frac{3}{2}\right) dt = 0$$

$$(d) \int_{-3}^2 [\exp(-t+1) + \sin\left(\frac{2\pi}{3}t\right)] \delta\left(t - \frac{3}{2}\right) dt = e^{-0.5} + \sin(\pi) = e^{-0.5}$$

$$(e) \int_{-\infty}^{\infty} \exp[-5t+1] \delta'(t-5) dt = -5e^{-5(5)} = -5e^{-25}$$

$$1.21 \quad P[X \leq x] = \int_{-\infty}^{x^+} f(x) dx$$

$$f(x) = 0.2 \delta(x+2) + 0.3 \delta(x) + 0.2 \delta(x-1) + 0.1 [u(x-3) - u(x-6)]$$



$$(a) \quad P[X \leq -3] = \int_{-\infty}^{-3} f(x) dx = 0$$

$$(b) \quad P[X \leq 1.5] = \int_{-\infty}^{1.5} f(x) dx = \int_{-\infty}^{1.5} [0.2 \delta(x+2) + 0.3 \delta(x) + 0.2 \delta(x-1)] dx$$

$$= 0.2 + 0.3 + 0.2 = 0.7$$

$$(c) \quad P[X \leq 4] = \int_{-\infty}^4 f(x) dx$$

$$= 0.2 + 0.3 + 0.2 + 0.1 \int_3^4 dx$$

$$= 0.2 + 0.3 + 0.2 + 0.1 = 0.8$$

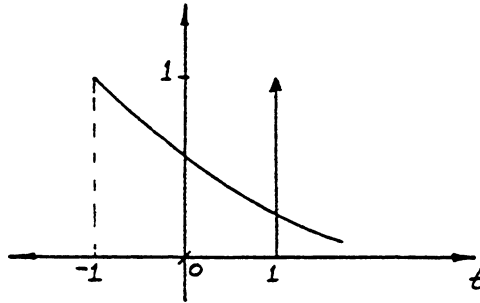
$$(d) \quad P[X \leq 6] = \int_{-\infty}^6 f(x) dx$$

$$= 0.2 + 0.3 + 0.2 + 0.1 \int_3^6 dx$$

$$= 0.2 + 0.3 + 0.2 + 0.3 = 1$$

1.22 $V(t) = e^{-(t+1)} u(t+1) + \delta(t-1)$

(a)



$$\begin{aligned}
 (b) \quad f(t) &= m \frac{d}{dt} [v(t)] \\
 &= \frac{d}{dt} [\exp[-(t+1)] u(t+1) + \delta(t-1)] \times 10^{-3} \\
 &= (\exp[-(t+1)] \delta(t+1) - \exp[-(t+1)] u(t+1) \\
 &\quad + \delta'(t-1)) \times 10^{-3} \\
 &= (\delta(t+1) - e^{-(t+1)} u(t+1) + \delta'(t-1)) \times 10^{-3} \text{ N}
 \end{aligned}$$

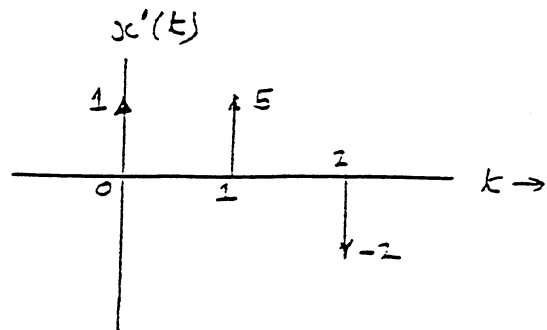
$$\begin{aligned}
 (c) \quad f_k(t) &= k \int_{-\infty}^t u(\tau) d\tau \\
 &= k \int_{-\infty}^t \exp[-(\tau+1)] u(\tau+1) + \delta(\tau-1) d\tau
 \end{aligned}$$

(1) $t < -1 \quad f_k(t) = 0$

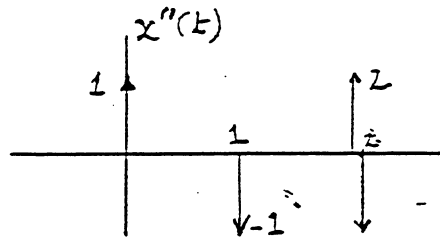
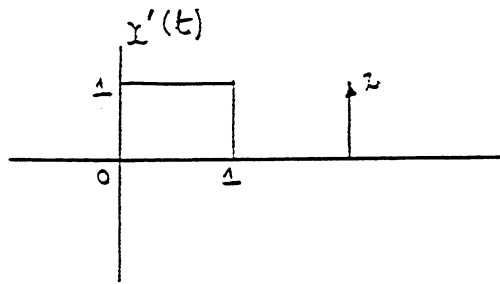
(2) $-1 < t < 1 \quad f_k(t) = 1 - \exp[-(t+1)] \text{ N}$

(3) $t > 1 \quad f_k(t) = 1 - \exp[-(t+1)] + u(t-1) \text{ N}$

1.23 (a) $x'(t) = \delta(t) + 5\delta(t-1) - 2\delta(t-2)$



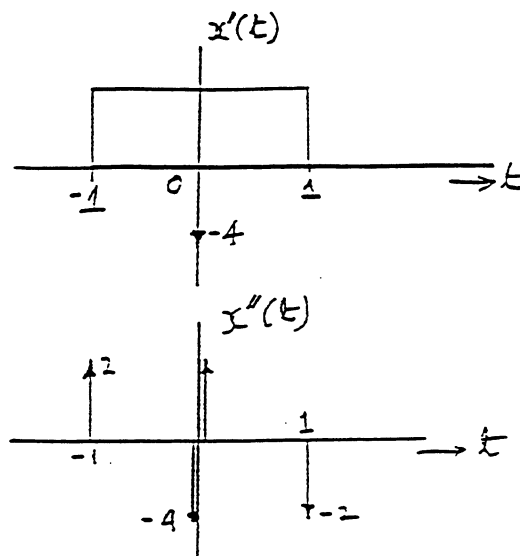
(b) $x'(t) = u(t) - u(t-1) + 2\delta(t-2)$
 $x''(t) = \delta(t) - \delta(t-1) + 2\delta'(t-2)$



(c) From Equation (1.12), we see that

$$x'(t) = 2u(t+1) - 4\delta(t) - 2u(t-1)$$

$$x''(t) = 2\delta(t+1) - 4\delta'(t) - 2\delta(t-1)$$



Chapter 2

- 2.1 (a) Nonlinear, causal, time-invariant, memoryless.
 (b) Nonlinear, causal, time-invariant, memoryless.
 (c) Linear, causal, time-invariant, memoryless.
 (d) Linear, causal, time-varying, memoryless.
 (e) Linear, causal, memoryless.

To check for time-invariance, let $x(t) = u(t)$ and let $x_1(t) = x(t+1)$. Then

$$y(t) = \begin{cases} 1 & t \geq 0 \\ 0 & t < 0 \end{cases}$$

$$y_1(t) = \begin{cases} x(t+1) & t \geq 0 \\ -x(t+1) & t < 0 \end{cases}$$

$$= \begin{cases} 1 & t \geq 0 \\ -1 & -1 \leq t < 0 \end{cases} \neq y(t+1)$$

so that the system is time-varying.

- (f) Linear, causal, with memory.

To check for time-invariance, let $x_1(t) = x(t - t_0)$. Then

$$y_1(t) = \int_{-\infty}^t x_1(\tau) d\tau = \int_{-\infty}^t x(\tau - t_0) d\tau$$

$$= \int_{-\infty}^{t-t_0} x(\tau) d\tau = y(t - t_0)$$

so that the system is time-invariant.

- (g) Linear, causal, time-invariant, with memory.

To check for time-invariance, let $x_1(t) = x(t - t_0)$. Then

$$y_1(t) = \int_0^t x_1(\tau) d\tau = \int_0^t x(\tau - t_0) d\tau$$

$$= \int_{-t_0}^{t-t_0} x(\tau) d\tau$$

But

$$y(t - t_0) = \int_0^{t-t_0} x(\tau) d\tau \neq y_1(t)$$

so that the system is time-varying.

(h) Linear, causal, time-invariant, with memory.

(i) Nonlinear, time-invariant, causal, memoryless.

(j) Nonlinear, time-invariant, causal, with memory.

(k) Linear, non-causal, time-varying, with memory.

(l) Linear if zero initial conditions, time-invariant, causal, with memory.

$$\begin{aligned} 2.2 \quad \int_{-\infty}^{\infty} x(\tau) \delta(t - \tau) d\tau &= \lim_{\Delta \rightarrow 0} \int_{-\infty}^{\infty} x(\tau) \frac{1}{\Delta} \text{rect}((t - \tau)/\Delta) d\tau \\ &= \lim_{\Delta \rightarrow 0} \frac{1}{\Delta} \int_{t - \frac{\Delta}{2}}^{t + \frac{\Delta}{2}} x(\tau) d\tau \end{aligned}$$

The integral is the area under $x(t)$ between

$t - \frac{\Delta}{2}$ and $t + \frac{\Delta}{2}$. This area is approximated by

$x(t)\Delta$, therefore

$$\int x(\tau) \delta(t - \tau) d\tau = \lim_{\Delta \rightarrow 0} \frac{1}{\Delta} x(t) \Delta = x(t)$$

2.3 Note that $u(t+a) * u(t+b) = (t+b+a) u(t+b+a)$

$$(a) \quad y(t) = \begin{cases} 1 & \frac{a}{2} + b < t < \frac{3a}{2} + b \\ 0 & \text{o.w.} \end{cases} = \text{rect}\left(\frac{t - a - b}{a}\right)$$

$$\begin{aligned}
 (b) \quad y(t) &= [u(t + \frac{a}{2}) - u(t - \frac{a}{2})] * [u(t + \frac{a}{2}) - u(t - \frac{a}{2})] \\
 &= (t + a) u(t + a) - t u(t) + (t - a) u(t - a) \\
 &= \begin{cases} t + a & -a \leq t \leq 0 \\ -t + a & 0 \leq t \leq a \\ 0 & \text{o.w.} \end{cases}
 \end{aligned}$$

$$\begin{aligned}
 (c) \quad y(t) &= \text{rect}(\frac{t}{a}) * u(t) = [u(t + \frac{a}{2}) - u(t - \frac{a}{2})] * u(t) \\
 &= (t + \frac{a}{2}) u(t + \frac{a}{2}) - (t - \frac{a}{2}) u(t - \frac{a}{2}) = \begin{cases} t + \frac{a}{2} & -\frac{a}{2} \leq t \leq \frac{a}{2} \\ a & t > \frac{a}{2} \\ 0 & t < -\frac{a}{2} \end{cases}
 \end{aligned}$$

$$(d) \quad y(t) = \text{rect}(\frac{t}{a}) * \text{sgn } t = \text{rect}(\frac{t}{a}) * [-u(-t) + u(t)]$$

$$\text{Now } x(t) * u(-t) = \int_t^\infty x(\tau) d\tau$$

$$\therefore y(t) = a \quad \text{all } t.$$

$$(e) \quad y(t) = u(t) * u(t) = t u(t) = r(t)$$

$$\begin{aligned}
 (f) \quad y(t) &= t [u(t) - u(t-1)] * u(t) \\
 &= t u(t) * u(t) - (t-1) u(t-1) * u(t) - u(t-1) * u(t) \\
 &= \frac{t^2}{2} u(t) - \frac{(t-1)^2}{2} u(t-1) - (t-1) u(t-1)
 \end{aligned}$$

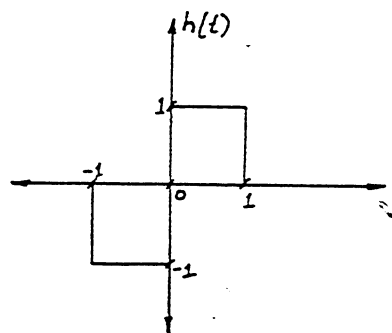
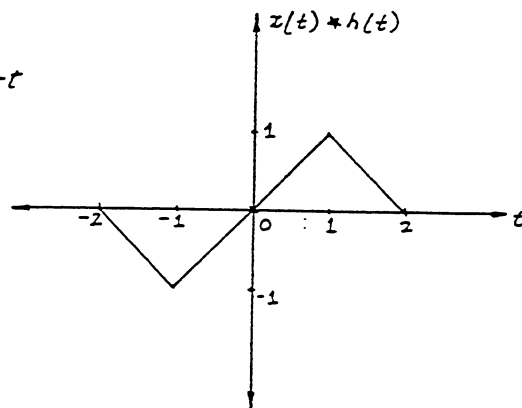
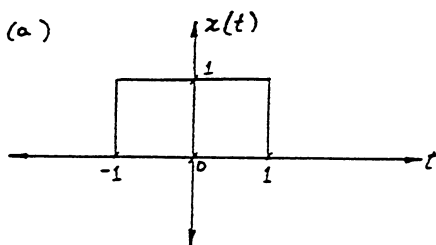
$$\begin{aligned}
 (g) \quad y(t) &= \text{rect}(\frac{t}{a}) * r(t) \\
 &= \begin{cases} 0 & t \leq -\frac{a}{2} \\ \int_{t-\frac{a}{2}}^{t+\frac{a}{2}} \tau d\tau & -\frac{a}{2} \leq t \\ 0 & t \geq \frac{a}{2} \end{cases} = \begin{cases} 0 & t \leq -\frac{a}{2} \\ a t & t \geq \frac{a}{2} \end{cases}
 \end{aligned}$$

$$\begin{aligned}
 (h) \quad y(t) &= x(t) * [\operatorname{sgn} t + u(-t-1)] = x(t) * [-u(t+1) + 2u(t)] \\
 &= -\left(\frac{t+1}{2}\right)^2 u(t+1) + t^2 u(t)
 \end{aligned}$$

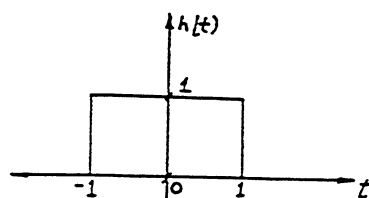
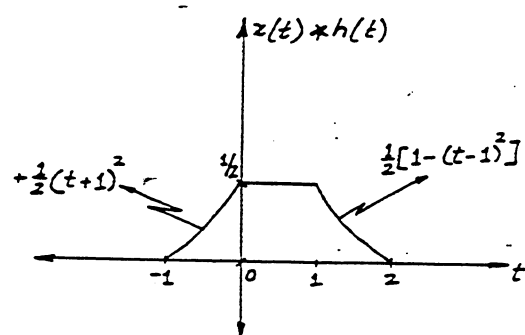
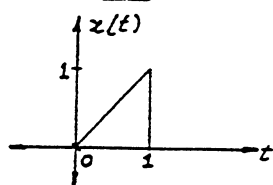
$$\begin{aligned}
 (i) \quad y(t) &= [u(t+1) - u(t-1)] \operatorname{sgn} t * u(t) \\
 &= [-u(t+1) + 2u(t) - u(t-1)] * u(t) \\
 &= \begin{cases} 0 & t \leq -1 \\ -t-1 & -1 \leq t \leq 0 \\ t-1 & 0 \leq t \leq 1 \\ 0 & t \geq 1 \end{cases}
 \end{aligned}$$

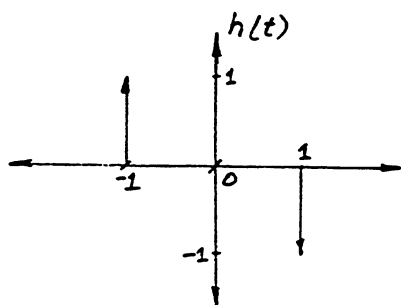
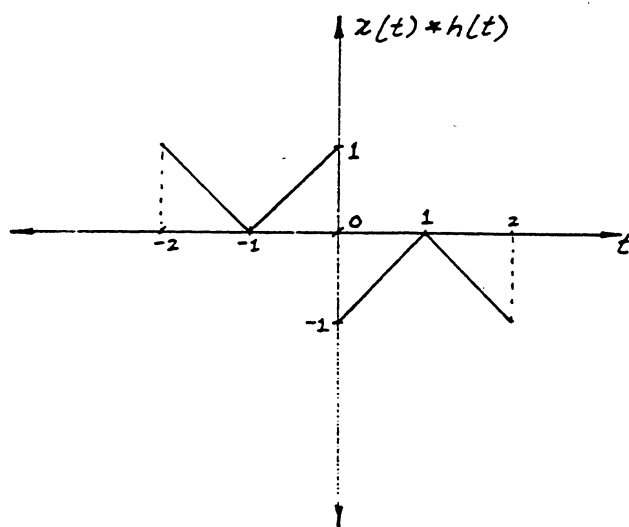
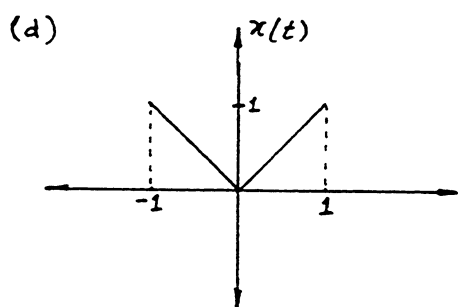
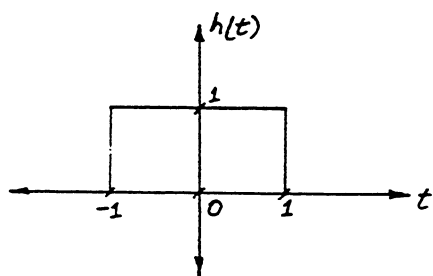
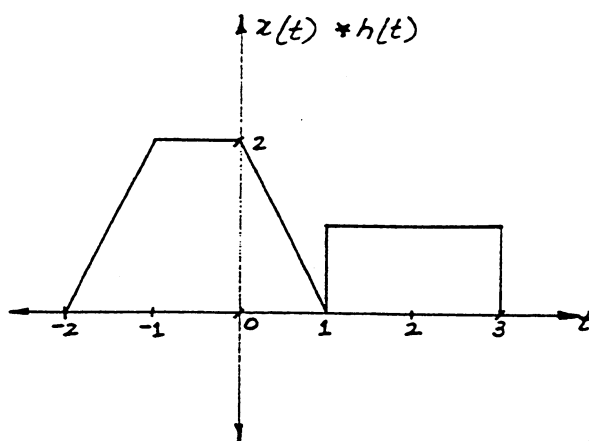
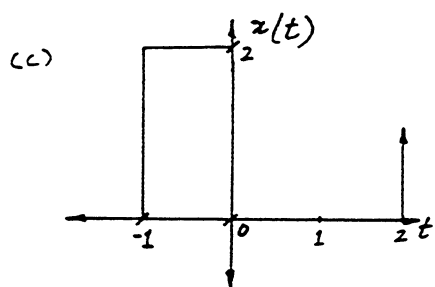
$$\begin{aligned}
 (j) \quad y(t) &= u(t) * \delta'(t) = \int_{-\infty}^{\infty} u(t-\tau) \delta'(\tau) d\tau \\
 &= \int_{-\infty}^t \delta'(\tau) d\tau = \delta(t)
 \end{aligned}$$

2.4 (a)



(b)





2.5 We can use the results of Example (2.3.4) to solve most of these problems.

$$(a) \quad y(t) = [e^{-t} - e^{-2t}] u(t)$$

$$(b) \quad y(t) = \int_0^t \tau e^{-\tau} d\tau = [1 - (t+1)e^{-t}] u(t)$$

$$(c) \quad y(t) = \int_0^t [e^{-\tau} + 1] d\tau = [1 - e^{-t} + t] u(t)$$

$$\begin{aligned} (d) \quad y(t) &= [\bar{e}^{-2t} u(t) + \delta(t)] * u(t) \\ &= \bar{e}^{-2t} u(t) * u(t) + u(t) \\ &= \frac{1}{2} (1 - \bar{e}^{-2t}) u(t) + u(t) \end{aligned}$$

$$\begin{aligned} (e) \quad y(t) &= \bar{e}^{-at} u(t) * [u(t) - \bar{e}^{-at} u(t-b)] \\ &= \bar{e}^{-at} u(t) * [u(t) - \bar{e}^{-ab} \bar{e}^{-a(t-b)} u(t-b)] \\ &= [1 - \bar{e}^{-at}] u(t) - (t-b) \bar{e}^{-at} u(t-b) \end{aligned}$$

$$\begin{aligned} (f) \quad y(t) &= [\delta(t-1) + \bar{e}^{-t}] u(t) * \bar{e}^{-2t} u(t) \\ &= \bar{e}^{-2(t-1)} u(t-1) + [e^{-t} - \bar{e}^{-2t}] u(t) \end{aligned}$$

$$\begin{aligned}
 2.6 \quad (a) \quad x(t) * y(-t) &= \int_{-\infty}^{\infty} x(\tau) y(-(t-\tau)) d\tau \\
 &= \int_{-\infty}^{\infty} x(\tau) y(\tau-t) d\tau = R_{xy}(t)
 \end{aligned}$$

$$\begin{aligned}
 (b) \quad R_{yx}(t) &= \int_{-\infty}^{\infty} y(\tau) x(\tau-t) d\tau \\
 &= \int_{-\infty}^{\infty} x(\tau') y(\tau'+t) d\tau' \neq R_{xy}(t)
 \end{aligned}$$

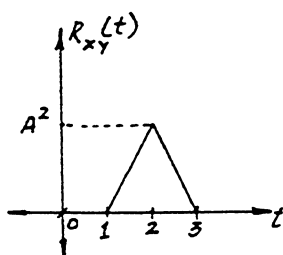
$$(c) \quad R_{yx}(-t) = \int_{-\infty}^{\infty} x(\tau') y(\tau'-t) d\tau' = R_{xy}(t)$$

2.7

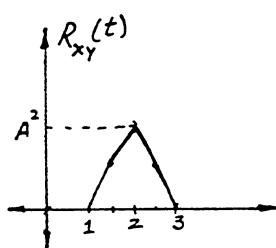
$$R_{xy}(t) = \begin{cases} 0 & t \leq 1 \\ A(A+B)(t-1) & 1 < t \leq \frac{3}{2} \\ \frac{A(A+B)}{2} + A(A-B)(t-\frac{3}{2}) & \frac{3}{2} < t \leq 2 \\ \frac{A(A-B)}{2} + A(A+B)(\frac{5}{2}-t) & 2 < t \leq \frac{5}{2} \\ A(A-B)(3-t) & \frac{5}{2} < t \leq 3 \\ 0 & t > 3 \end{cases}$$

$$R_{xy}(t) = \begin{cases} 0 & t \leq 1 \\ A^2 \left(1 + \frac{B}{A}\right) (t-1) & 1 < t \leq \frac{3}{2} \\ \frac{A^2}{2} \left(1 + \frac{B}{A}\right) + A^2 \left(1 - \frac{B}{A}\right) \left(t - \frac{3}{2}\right) & \frac{3}{2} < t \leq 2 \\ \frac{A^2}{2} \left(1 - \frac{B}{A}\right) + A^2 \left(1 + \frac{B}{A}\right) \left(\frac{5}{2} - t\right) & 2 < t \leq \frac{5}{2} \\ A^2 \left(1 - \frac{B}{A}\right) (3 - t) & \frac{5}{2} < t \leq 3 \\ 0 & t > 3 \end{cases}$$

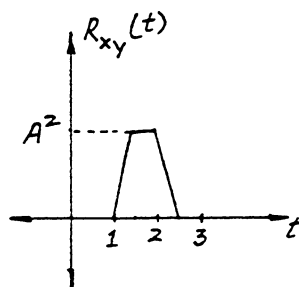
$$\frac{B}{A} = 0.0$$



$$\frac{B}{A} = 0.1$$



$$\frac{B}{A} = 1$$



$$2.8 \text{ (a)} \quad R_x(t) = \int_{-\infty}^{\infty} x(\tau) x(\tau+t) d\tau$$

$$R_x(0) = \int_{-\infty}^{\infty} x^2(t) dt = E$$

$$(b) \quad R_x(t) \leq \left(\int_{-\infty}^{\infty} x^2(\tau) d\tau \int_{-\infty}^{\infty} x^2(\tau+t) d\tau \right)^{1/2}$$

$$= \left(\int_{-\infty}^{\infty} x^2(\tau) d\tau \int_{-\infty}^{\infty} x^2(\tau') d\tau' \right)^{1/2}$$

$$= R(0)$$

Q.E.D.

$$\begin{aligned}
(c) \quad R_z(t) &= \int_{-\infty}^{\infty} z(\tau) z(\tau+t) d\tau \\
&= \int_{-\infty}^{\infty} (x(\tau) + y(\tau)) (x(\tau+t) + y(\tau+t)) d\tau \\
&= R_x(t) + R_{yx}(t) + R_{xy}(t) + R_y(t)
\end{aligned}$$

$$\begin{aligned}
2.9 \quad R_y(t) &= \int_{-\infty}^{\infty} y(\tau) y(\tau+t) d\tau \\
&= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x(\tau-\mu) h(\mu) d\mu \int_{-\infty}^{\infty} x(\tau+t-\nu) h(\nu) d\nu d\tau \\
&= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(\mu) h(\nu) \int_{-\infty}^{\infty} x(\tau-\mu) x(\tau+t-\nu) d\tau d\mu d\nu \\
&= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(\mu) h(\nu) R_x(t-\nu+\mu) d\mu d\nu \\
&= \int_{-\infty}^{\infty} h(\mu) G(t+\mu) d\mu
\end{aligned}$$

$$\begin{aligned}
\text{where } G(\eta) &= \int_{-\infty}^{\infty} h(\nu) R_x(\eta-\nu) d\nu \\
&= R_x(t) * h(t)
\end{aligned}$$

but the last integral is $h(-t) * G(t)$
therefore

$$R_x(t) = R_x(t) * h(t) * h(-t)$$

$$2.10 \quad y(t) = x(t) * h(t)$$

$$= \int_{-\infty}^{\infty} h(\tau) x(t-\tau) d\tau$$

$$= \int_{-\infty}^{\infty} h(\tau) \exp[j\omega(t-\tau)] d\tau$$

$$= \exp[j\omega t] \int_{-\infty}^{\infty} h(\tau) \exp[-j\omega\tau] d\tau$$

$$= \exp[j\omega t] H(\omega)$$

$$2.11 \quad (a) \quad h(t) = e^{-3t} \sin t u(t), \text{ Causal (C), stable (S).}$$

$$(b) \quad h(t) = e^{4t} u(-t), \text{ Noncausal (NC), S}$$

$$(c) \quad h(t) = -t e^{-t} u(-t), \text{ NC, unstable (u.s.)}$$

$$(d) \quad h(t) = e^{-1/|t|}, \text{ NC, S}$$

$$(e) \quad h(t) = |t-1| e^{-2|t|}, \text{ NC, S}$$

$$(f) \quad h(t) = \text{rect}\left(\frac{t}{2}\right), \text{ NC, S}$$

$$(g) \quad h(t) = \delta(t) + e^{-3t} u(t), \text{ C, S}$$

$$(h) \quad h(t) = \delta'(t) + e^{-1t}, \text{ N.C., u.s.}$$

$$(i) \quad h(t) = \delta'(t) + e^{-2|t|}, \text{ N.C., u.s.}$$

$$(j) \quad h(t) = (1-t) \text{rect}(t/3), \text{ NC, S}$$

2.12 (a) $h(t) = \delta(t+2)$ is invertible with inverse

$$h_I(t) = \delta(t-2)$$

(b) $h(t) = u(t)$ is invertible with inverse

$$h_I(t) = \delta'(t)$$

(c) $h(t) = \delta(t-3)$ is invertible with inverse

$$h_I(t) = \delta(t+3)$$

(d) $h(t) = \text{rect}(t/4)$ is not invertible. The two

inputs $x_1(t) = \sin \frac{\pi}{2} t$ and $x_2 = \sin \pi t$

produce the same output $y(t) = 0$

(e) $h(t) = \exp[-t] u(t)$ is invertible and the inverse

$$\text{system is } h_I(t) = \delta'(t) + \delta(t)$$

2.13 (a) No, this would contradict the assumption that system I is the optimum operation on $X(t)$.

(b) No, this would contradict the statement that the second operation in system III is optimum.

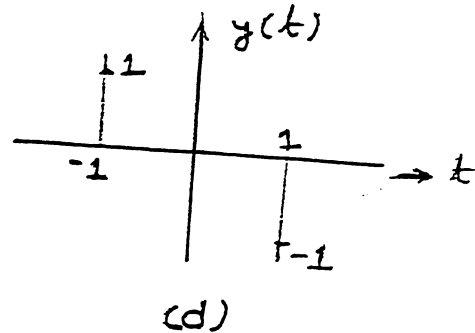
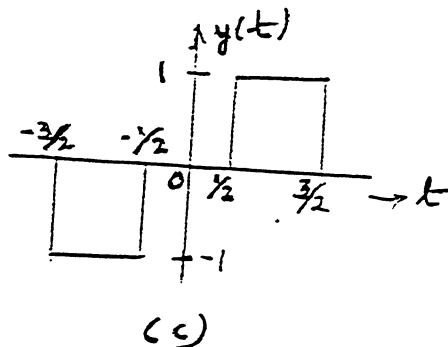
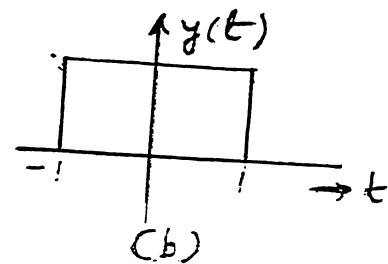
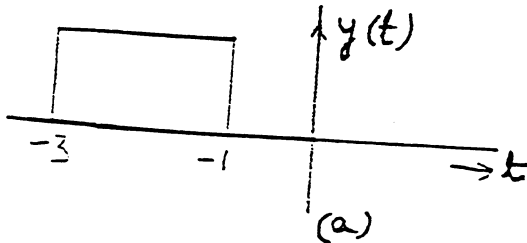
(c) These systems are identical from the performance view point.

(d) No, need only the system to be invertible.

$$\begin{aligned}
 2.14 \quad h(t) &= h_5(t) * [h_1(t) + h_2(t) * h_3(t) + h_4(t)] \\
 &= e^{-3t} u(t) * [\bar{e}^{-2t} u(t) + [\bar{e}^{-t} - \bar{e}^{-2t}] u(t) + \delta(t)] \\
 &= e^{-3t} u(t) * [\bar{e}^{-t} u(t) + \delta(t)] \\
 &= \frac{1}{t} [\bar{e}^{-t} - \bar{e}^{-3t}] u(t) + \bar{e}^{-3t} u(t) \\
 &= \frac{1}{t} [\bar{e}^{-t} + \bar{e}^{-3t}] u(t)
 \end{aligned}$$

stable

2.15



$$2.16 \quad \frac{L}{R} h'(t) + h(t) = \delta(t)$$

$$h(t) = \frac{R}{L} \exp[-Rt/L] u(t)$$

$$2.17 \quad RC \frac{dh(t)}{dt} + h(t) = \delta(t)$$

$$(a) \text{ If } h(t) = \frac{1}{RC} \exp[-t/RC] u(t)$$

$$x(t) = u(t - \tau/2)$$

(b) If $y(t) = [1 - \exp[-(t - \theta/2)/RC]] u(t - \theta/2)$

$$x(t) = u(t + \tau/2)$$

$$y(t) = (1 - \exp[-(t + \theta/2)/RC]) u(t + \theta/2)$$

(c) If $x(t) = \text{rect}(t/\tau) = y(t + \tau/2) - u(t - \tau/2)$

$$\therefore y(t) = \text{rect}(\theta/2) - \exp[-(t + \theta/2)/RC] u(t + \theta/2) + \exp[-(t - \theta/2)/RC] u(t - \theta/2)$$

2.18 $\frac{dh(t)}{dt} + \frac{h(t)}{RC} = \delta'(t)$

$$h(t) = \delta(t) - \exp(-t/RC) u(t)$$

(a) If $x(t) = u(t - \theta/2)$

$$y(t) = u(t - \theta/2) + [RC \exp[-(t - \theta/2)/RC] - 1] u(t - \theta/2) = RC \exp[-(t - \theta/2)/RC] u(t - \theta/2)$$

(b) If $x(t) = u(t + \theta/2)$

$$y(t) = RC \exp[-(t + \theta/2)/RC] u(t + \theta/2)$$

(c) If $x(t) = \text{rect}(t/\theta) = u(t + \theta/2) - u(t - \theta/2)$

$$y(t) = RC \exp[-(t + \theta/2)/RC] u(t + \theta/2) - RC \exp[-(t - \theta/2)/RC] u(t - \theta/2)$$

2.19 Let $x_1(t)$ and $x_2(t)$ be two input signals, and let $y_1(t)$ and $y_2(t)$ be the corresponding responses.

That is

$$\frac{d^N y_1(t)}{dt^N} + \sum_{k=0}^{N-1} a_k(t) \frac{d^k y_1(t)}{dt^k} = \sum_{k=0}^M b_k(t) \frac{d^k x_1(t)}{dt^k} \quad (1)$$

$$\frac{d^N y_2(t)}{dt^N} + \sum_{k=0}^{N-1} a_k(t) \frac{d^k y_2(t)}{dt^k} = \sum_{k=0}^M b_k(t) \frac{d^k x_2(t)}{dt^k} \quad (2)$$

and also $y_1(t)$ and $y_2(t)$ must satisfy

$$y_1^{(k)}(t_0) = y_2^{(k)}(t_0) = 0 \quad (3)$$

Consider next the input $x_3(t) = \alpha x_1(t) + \beta x_2(t)$, where α and β are any complex numbers. Then,

using eqns (1) and (2), it is not difficult to see that $y_3(t) = \alpha y_1(t) + \beta y_2(t)$ satisfies the differential equation

$$\frac{d^N y_3(t)}{dt^N} + \sum_{k=0}^{N-1} a_k(t) \frac{d^k y_3(t)}{dt^k} = \sum_{k=0}^M b_k(t) \frac{d^k x_3(t)}{dt^k}$$

and also, from eq. (3)

$$y_3^{(k)}(t_0) = \alpha y_1^{(k)}(t_0) + \beta y_2^{(k)}(t_0) = 0$$

Therefore, $y_3(t)$ is the response to $x_3(t)$, and thus the system is linear.

2.20 Let $y_1(t)$ be the response to an input $x_1(t)$, which is zero for $t < t_0$. That is

$$\frac{d^N y_1(t)}{dt^N} + \sum_{k=0}^{N-1} a_k \frac{d^k y_1(t)}{dt^k} = \sum_{k=0}^M b_k \frac{d^k x_1(t)}{dt^k} \quad (1)$$

$$y_1^{(k)}(t) = 0 \quad (2)$$

Now consider the input

$$x_2(t) = x_1(t - \tau) \quad (3)$$

note that $x_2(t)$ is zero for $t < t_0 + \tau$. Therefore, the response $y_2(t)$ to this input must satisfy the differential equation

$$\frac{d^N y_2(t)}{dt^N} + \sum_{k=0}^{N-1} a_k \frac{d^k y_2(t)}{dt^k} = \sum_{k=0}^M b_k \frac{d^k x_2(t)}{dt^k} \quad (4)$$

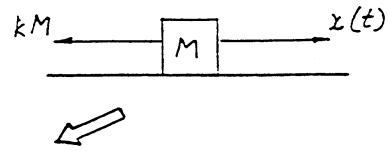
with the initial conditions

$$y_2^{(k)}(t_0 + \tau) = 0 \quad (5)$$

Using eqns. (1) and (2), it is straightforward to show that $y_1(t - \tau)$ satisfies eqns. (4) and (5),

and thus that

$$y_2(t) = y_1(t - \tau)$$



$$2.21 \quad M \frac{d^2 y(t)}{dt^2} = x(t) - kM$$

$$\text{or } M \frac{d^2 y(t)}{dt^2} + kM = x(t)$$

Assuming that $y(t_0) = y'(t_0) = y''(t_0) = 0$

then according to problem 2.19, the system is linear, also according to problem 2.20, the system is time-invariant. This system can be time-varying if- either k or M is changing with time.

$$2.22 \quad Ml\theta''(t) = x(t) - Mg \sin(\theta(t))$$

$$\theta''(t) + \frac{g}{l} \sin(\theta(t)) = \frac{x(t)}{Ml}$$

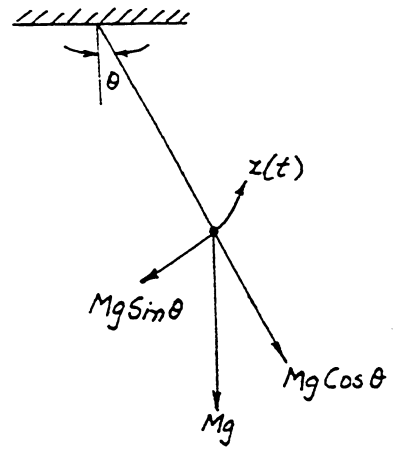
No, this system is not linear

If $\theta(t)$ is small such that

$$\sin(\theta(t)) \approx \theta(t), \text{ then}$$

$$\theta''(t) + \frac{g}{l} \theta(t) = \frac{x(t)}{Ml}$$

which is a linear system.



2.23

With reference to the figure, let $y_1 = u_1'' + u_1'$. Then

$$u_1'' = x - 2u_1' - u_1$$

$$\begin{aligned} y_1 &= 2u_1'' + u_1' = 2[x - 2u_1' - u_1] + u_1' \\ &= 2x - 3u_1' - 2u_1 \quad (a) \end{aligned}$$

$$\begin{aligned} y_1' &= 2x' - 3u_1'' - 2u_1' = 2x' - 3[x - 2u_1' - u_1] - 2u_1' \\ &= 2x' - 3x + 4u_1' + 3u_1 \quad (b) \end{aligned}$$

$$\begin{aligned} u_1'' &= 2x'' - 3x' + 4u_1' + 3u_1 \\ &= 2x'' - 3x' + 4[x - 2u_1' - u_1] + 3u_1 \quad (c) \\ &= 2x'' - 3x' + 4x - 5u_1' - 4u_1 \end{aligned}$$

$\therefore y_1'' + 2y_1' + y_1 = 2x'' + x'$ by adding (a), 2(b), (c).

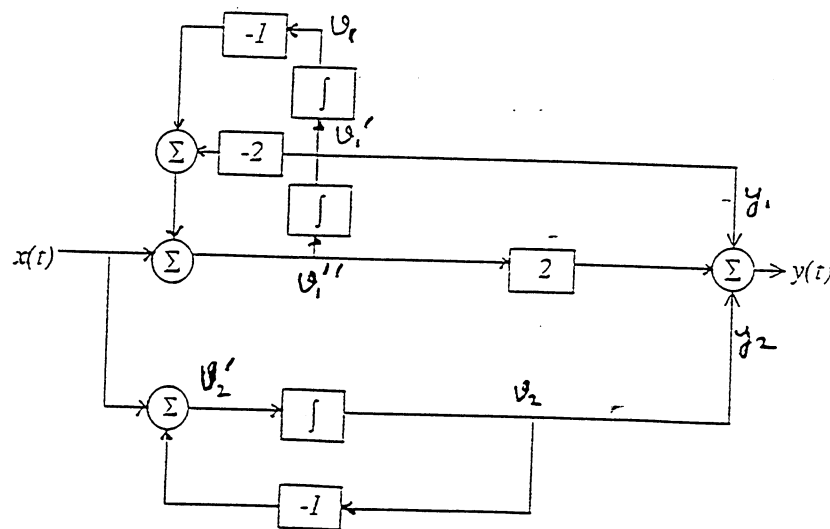
$$\text{Let } y_2 = u_2 \quad (d), \quad y_2' = x - u_2 = x - y_2 \quad (e)$$

$$\therefore y_2'' = x' - y_2' = x' - x + y_2 \quad (f)$$

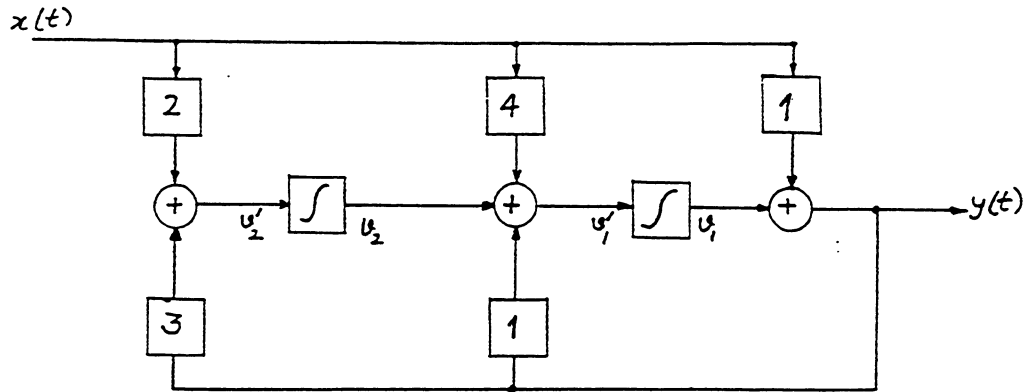
$\therefore y_2'' + 2y_2' + y_2 = x' + x$ by adding (d), 2(e), (f).

$$\therefore (y_1 + y_2)'' + 2(y_1 + y_2)' + (y_1 + y_2) = 2x'' + 2x' + x$$

$$\text{or } y'' + 2y' + y = 2x'' + 2x' + x$$



2.24



$$v_2'(t) = 2x(t) + 3y(t) \quad (1)$$

$$v_1'(t) = 4x(t) + y(t) + v_2(t) \quad (2)$$

$$y(t) = v_1(t) + x(t) \quad (3)$$

From (1), (2) and (3)

$$\begin{aligned} y(t) &= \int (4x(t) + y(t) + v_2(t)) dt + x(t) \\ &= \int (4x(t) + y(t) + \int (2x(t) + 3y(t)) dt) dt + x(t) \end{aligned}$$

or

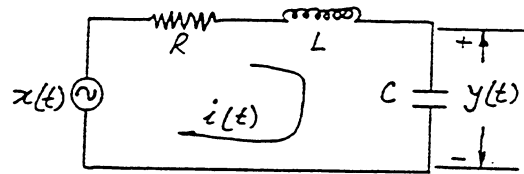
$$y'(t) = 4x'(t) + y'(t) + \int (2x(t) + 3y(t)) dt + x'(t)$$

$$y''(t) = 4x'(t) + y'(t) + 2x(t) + 3y(t) + x''(t)$$

Rearranging terms, yields

$$y''(t) - y'(t) - 3y(t) = 2x(t) + 4x'(t) + x''(t)$$

2.25 (a)



$$V_L(t) = L \frac{di(t)}{dt} \quad i = C \frac{dy(t)}{dt}$$

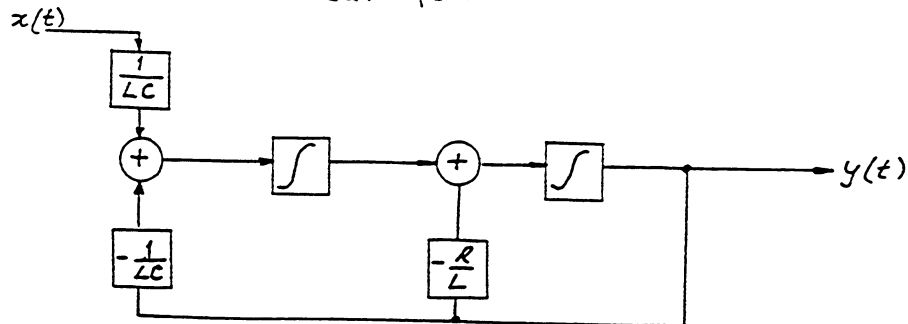
$$x(t) = R i(t) + L \frac{di(t)}{dt} + y(t)$$

$$\text{or } x(t) = LC \frac{d^2 y(t)}{dt^2} + RC \frac{dy(t)}{dt} + y(t)$$

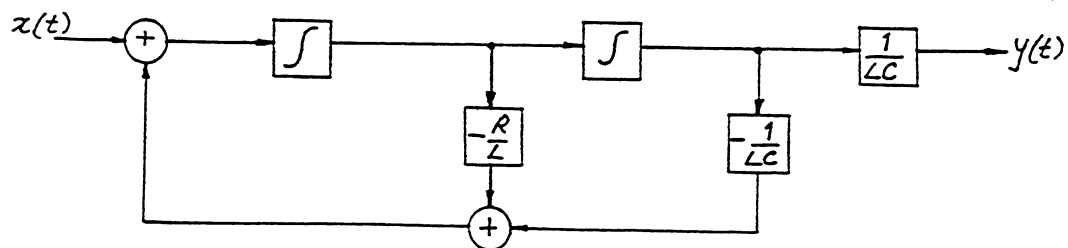
$$\Rightarrow y''(t) + \frac{R}{L} y'(t) + \frac{1}{LC} y(t) = \frac{1}{LC} x(t)$$

$$a_0 = \frac{1}{LC}, \quad a_1 = \frac{R}{L}, \quad b_0 = \frac{1}{LC}$$

(b) first canonical form

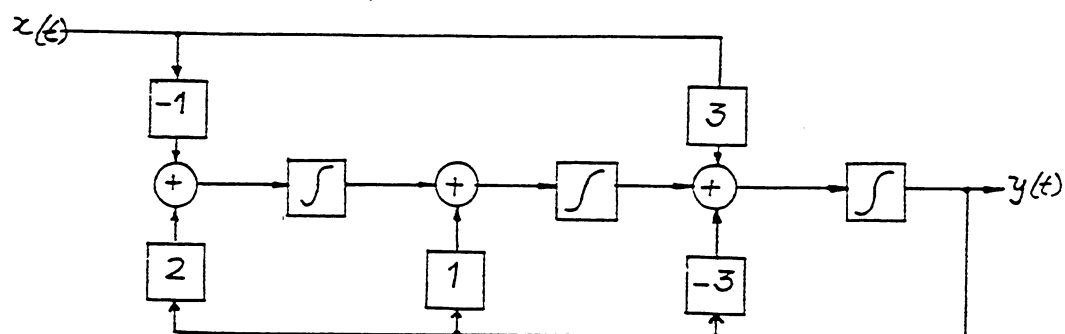


(c) second canonical form

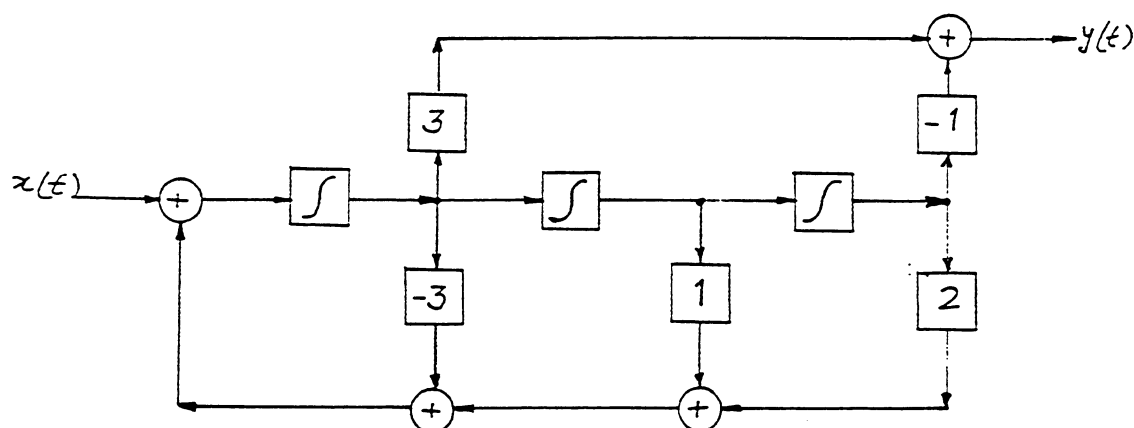


2.26 $a_2 = 3$, $a_1 = -1$, $a_0 = -2$, $b_2 = 3$, $b_1 = 0$, $b_0 = -1$

first canonical form



second canonical form



2.27 $i_C(t) = C \frac{dV_C(t)}{dt}$ $V_R(t) = R i_C(t) = y(t)$

$$x(t) = V_C(t) + y(t) = y(t) + \frac{1}{RC} \int y(t) dt$$

$$\Rightarrow y'(t) + \frac{1}{RC} y(t) = x'(t)$$

$$h'(t) + \frac{1}{RC} h(t) = \delta'(t) \quad \text{let } h_p(t) = D \delta(t)$$

$$\Rightarrow h(t) = B e^{-\frac{t}{RC}} u(t) + D \delta(t)$$

$$\text{or } B \delta(t) + D \delta'(t) + \frac{D}{RC} \delta(t) = \delta'(t) \Rightarrow D = 1, B = -\frac{1}{RC}$$

$$\Rightarrow h(t) = \delta(t) - \frac{1}{RC} e^{-\frac{t}{RC}} u(t)$$

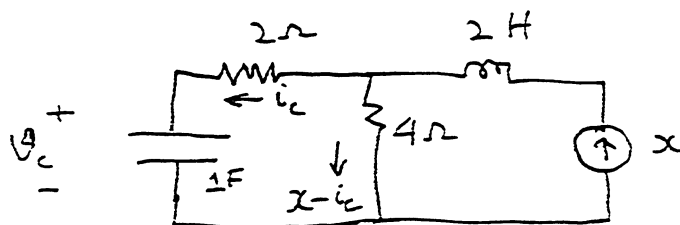
2.28 The first form is $\underline{v}'(t) = \begin{bmatrix} -\frac{5}{2} & 1 \\ -1 & 0 \end{bmatrix} \underline{v}(t) + \begin{bmatrix} 1 \\ 1 \end{bmatrix} x(t)$

$$y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} \underline{v}(t)$$

Second form is $\underline{v}'(t) = \begin{bmatrix} 0 & 1 \\ -1 & -\frac{5}{2} \end{bmatrix} \underline{v}(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} x(t)$

$$y(t) = \begin{bmatrix} 1 & 1 \end{bmatrix} \underline{v}(t)$$

2.29



Going around the first loop, we get

$$v_c - 2i_c = 4(x - i_c)$$

$$\text{Also } v_c = \frac{1}{C} \int i_c dt \quad \therefore i_c = \dot{v}_c$$

$$\therefore 6\dot{v}_c = v_c - 4x \quad \text{or } \dot{v}_c = \frac{1}{6}v_c - \frac{2}{3}x$$

$i_c = x$ and is not a state variable.

2.30

First form is $\underline{v}' = \begin{bmatrix} -1 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & 0 & 0 \end{bmatrix} \underline{v} + \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix} x$

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \underline{v}$$

Second form is

$$\underline{v}' = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & 0 & -1 \end{bmatrix} \underline{v} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} x$$

$$y = \begin{bmatrix} 1 & 1 \end{bmatrix} \underline{v}$$

$$2.31 \text{ (a)} \quad A = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -3 \end{bmatrix}$$

(i) Series expansion

$$e^{At} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} -t & 0 & 0 \\ 0 & -2t & 0 \\ 0 & 0 & -3t \end{bmatrix} + \begin{bmatrix} \frac{t^2}{2} & 0 & 0 \\ 0 & 2t^2 & 0 \\ 0 & 0 & \frac{9t^2}{2} \end{bmatrix} + \dots$$

$$= \begin{bmatrix} 1 - t + \frac{t^2}{2} + \dots & 0 & 0 \\ 0 & 1 - 2t + 2t^2 + \dots & 0 \\ 0 & 0 & 1 - 3t + \frac{9}{2}t^2 + \dots \end{bmatrix}$$

$$= \begin{bmatrix} e^{-t} & 0 & 0 \\ 0 & e^{-2t} & 0 \\ 0 & 0 & e^{-3t} \end{bmatrix}$$

(ii) Cayley - Hamilton

$$\lambda_1 = -1, \quad \lambda_2 = -2, \quad \lambda_3 = -3$$

$$\begin{bmatrix} e^{-t} \\ e^{-2t} \\ e^{-3t} \end{bmatrix} = \begin{bmatrix} y_0 & -y_1 & +y_2 \\ y_0 & -2y_1 & +4y_2 \\ y_0 & -3y_1 & +9y_2 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 1 \\ 1 & -2 & 4 \\ 1 & -3 & 9 \end{bmatrix} \begin{bmatrix} y_0 \\ y_1 \\ y_2 \end{bmatrix}$$

$$\begin{bmatrix} y_0 \\ y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 3 & -3 & 1 \\ \frac{5}{2} & -4 & \frac{3}{2} \\ \frac{1}{2} & -1 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} e^{-t} \\ e^{-2t} \\ e^{-3t} \end{bmatrix} = \begin{bmatrix} 3e^{-t} - 3e^{-2t} + e^{-3t} \\ \frac{5}{2}e^{-t} - 4e^{-2t} + \frac{3}{2}e^{-3t} \\ \frac{1}{2}e^{-t} - e^{-2t} + \frac{1}{2}e^{-3t} \end{bmatrix}$$

$$\exp At = y_0 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + y_1 \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -3 \end{bmatrix} + y_2 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 9 \end{bmatrix}$$

$$\exp At = \begin{bmatrix} e^{-t} & 0 & 0 \\ 0 & e^{-2t} & 0 \\ 0 & 0 & e^{-3t} \end{bmatrix}$$

$$(b) \quad A = \begin{bmatrix} -1 & 2 & -1 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \quad A^2 = \begin{bmatrix} 1 & -4 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad A^3 = \begin{bmatrix} -1 & 6 & -3 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$$A^4 = \begin{bmatrix} 1 & -8 & 4 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(i) Series expansion

$$\exp At = I + At + \frac{A^2 t^2}{2!} + \frac{A^3 t^3}{3!} + \frac{A^4 t^4}{4!} + \dots$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} -1 & 2 & -1 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} t + \begin{bmatrix} 1 & -4 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \frac{t^2}{2!} + \begin{bmatrix} -1 & 6 & -3 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \frac{t^3}{3!} + \dots$$

$$= \begin{bmatrix} 1 - t + \frac{t^2}{2!} - \frac{t^3}{3!} + \dots & 2t - \frac{4}{2}t^2 + \frac{6}{3!}t^3 + \dots & -t + \frac{2t^2}{2!} - \frac{3t^3}{3!} + \dots \\ 0 & 1 - t + \frac{t^2}{2} - \frac{t^3}{3!} + \dots & 0 \\ 0 & 0 & 1 - t + \frac{t^2}{2!} - \frac{t^3}{3!} + \dots \end{bmatrix}$$

$$= \begin{bmatrix} e^{-t} & 2t(e^{-t}) & -t(e^{-t}) \\ 0 & e^{-t} & 0 \\ 0 & 0 & e^{-t} \end{bmatrix}$$

(ii) Cayley - Hamilton

the eigenvalues of the matrix A are -1, -1, -1

$$e^{At} = \gamma_0 I + \gamma_1 A + \gamma_2 A^2$$

$$e^{-t} = \gamma_0 - \gamma_1 + \gamma_2$$

$$te^{-t} = \gamma_1 - 2\gamma_2$$

$$t^2 e^{-t} = 2 \gamma_2$$

$$\Rightarrow \gamma_2 = \frac{t^2}{2} e^{-t}$$

$$\gamma_1 = t e^{-t} + t^2 e^{-t}$$

$$\gamma_0 = e^{-t} + t e^{-t} + t^2 e^{-t} - \frac{t^2}{2} e^{-t} = e^{-t} + t e^{-t} + \frac{t^2}{2} e^{-t}$$

$$\exp At = \gamma_0 I + \gamma_1 A + \gamma_2 A^2$$

$$= \begin{bmatrix} e^{-t} & 2t e^{-t} & -t e^{-t} \\ 0 & e^{-t} & 0 \\ 0 & 0 & e^{-t} \end{bmatrix}$$

$$(c) \quad A = \begin{bmatrix} -1 & 1 & -1 \\ 0 & 1 & -4 \\ 0 & 1 & -3 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 1 & -1 & 0 \\ 0 & -3 & 8 \\ 0 & -2 & 5 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} -1 & 0 & 3 \\ 0 & 5 & -12 \\ 0 & 3 & -7 \end{bmatrix}$$

(i) Series expansion

$$\exp At = I + At + \frac{A^2 t^2}{2!} + \frac{A^3 t^3}{3!} + \dots$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} -1 & 1 & -1 \\ 0 & 1 & -4 \\ 0 & 1 & -3 \end{bmatrix} t + \begin{bmatrix} 1 & -1 & 0 \\ 0 & -3 & 8 \\ 0 & -2 & 5 \end{bmatrix} \frac{t^2}{2!}$$

$$+ \begin{bmatrix} -1 & 0 & 3 \\ 0 & 5 & -12 \\ 0 & 3 & -7 \end{bmatrix} \frac{t^3}{3!} + \dots$$

$$= \begin{bmatrix} 1 - t + \frac{t^2}{2} + \dots & t - \frac{t^2}{2!} + \dots & -t + \frac{3t^3}{3!} + \dots \\ 0 & 1 + t - \frac{3t^2}{2!} + \frac{5t^3}{3!} + \dots & -4t + \frac{8t^2}{2!} - \frac{12}{3!}t^3 \dots \\ 0 & t - \frac{2t^2}{2!} + \frac{3t^3}{3!} + \dots & 1 - 3t + \frac{5t^2}{2} - \frac{7}{3!}t^3 \dots \end{bmatrix}$$

$$= \begin{bmatrix} e^{-t} & (t + \frac{t^2}{2})e^{-t} & (t + t^2)e^{-t} \\ 0 & (1 + 2t)e^{-t} & -4te^{-t} \\ 0 & te^{-t} & (1 - 2t)e^{-t} \end{bmatrix}$$

(iii) Cayley - Hamilton

the eigenvalues of $A = -1, -1, -1$

$$e^{At} = \gamma_0 + \gamma_1 \lambda + \gamma_2 \lambda^2, \text{ for } \lambda_i = -1$$

$$\Rightarrow e^{-t} = \gamma_0 - \gamma_1 + \gamma_2$$

$$te^{-t} = \gamma_1 - 2\gamma_2$$

$$t^2 e^{-t} = 2\gamma_2 \Rightarrow \gamma_2 = \frac{t^2}{2} e^{-t}$$

$$\gamma_1 = te^{-t} + t^2 e^{-t} = (t + t^2)e^{-t}$$

$$\gamma_0 = e^{-t} + te^{-t} + t^2 e^{-t} - \frac{t^2}{2} e^{-t} = (1 + t + \frac{t^2}{2})e^{-t}$$

$$\exp(At) = \gamma_0 I + \gamma_1 A + \gamma_2 A^2$$

$$= \begin{bmatrix} e^{-t} & (t + \frac{t^2}{2})e^{-t} & (t + t^2)e^{-t} \\ 0 & (1 + 2t)e^{-t} & -4te^{-t} \\ 0 & te^{-t} & (1 - 2t)e^{-t} \end{bmatrix}$$

2-32 Second canonical form gives $\underline{v}' = \begin{bmatrix} 0 & 1 \\ -8 & -6 \end{bmatrix} \underline{v} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} x$

$$y = \begin{bmatrix} 1 & 1 \end{bmatrix} \underline{v}$$

Eigen values of A are $\lambda_1 = -2, \lambda_2 = -4$

Write $e^{-2t} = r_0 + r_1(-2)$

$$e^{-4t} = r_0 + r_1(-4)$$

to get $r_0 = \frac{1}{2} e^{-2t} - \frac{1}{2} e^{-4t}$

$$r_1 = \frac{1}{2} [e^{-2t} - e^{-4t}]$$

$$\therefore \underline{\Phi}(t) = e^{At} = r_0(t)I + r_1(t)A$$

$$= (2e^{-2t} - e^{-4t}) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \frac{1}{2} (e^{-2t} - e^{-4t}) \begin{bmatrix} 0 & 1 \\ -8 & -6 \end{bmatrix}$$

$$= \begin{bmatrix} 2e^{-2t} - e^{-4t} & \frac{1}{2}e^{-2t} - \frac{1}{2}e^{-4t} \\ -4e^{-2t} + 4e^{-4t} & -e^{-2t} + 2e^{-4t} \end{bmatrix}$$

$$h(t) = c \underline{\Phi} b = -\frac{1}{2} e^{-2t} + \frac{3}{2} e^{-4t} \quad t \geq 0$$

2-33 $y''(t) + 7y'(t) + 12y(t) = x''(t) - 3x'(t) + 4x(t)$

$$y'(0) = y(0) = 0 \quad x(t) = \delta(t)$$

the second canonical form of the system is given by

$$\begin{bmatrix} v_1'(t) \\ v_2'(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -12 & -7 \end{bmatrix} \begin{bmatrix} v_1(t) \\ v_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} x(t)$$

$$y(t) = \begin{bmatrix} -8 & -10 \end{bmatrix} \begin{bmatrix} v_1(t) \\ v_2(t) \end{bmatrix} + x(t)$$

where $A = \begin{bmatrix} 0 & 1 \\ -12 & -7 \end{bmatrix}$, $b = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $c = \begin{bmatrix} -8 & -10 \end{bmatrix}$, $d = 1$

the eigenvalues of A are $-3, -4$

$$\begin{cases} e^{-3t} = \gamma_0 - 3\gamma_1 \\ e^{-4t} = \gamma_0 - 4\gamma_1 \end{cases} \Rightarrow \begin{cases} \gamma_0 = 4e^{-3t} - 3e^{-4t} \\ \gamma_1 = e^{-3t} - e^{-4t} \end{cases}$$

$$\exp At = \gamma_0 I + \gamma_1 A = \begin{bmatrix} 4e^{-3t} - 3e^{-4t} & e^{-3t} - e^{-4t} \\ -12e^{-3t} + 12e^{-4t} & -3e^{-3t} + 4e^{-4t} \end{bmatrix}$$

$$h(t) = C \exp(At) b + d \delta(t)$$

$$= \begin{bmatrix} -8 & -10 \end{bmatrix} \begin{bmatrix} 4e^{-3t} - 3e^{-4t} & e^{-3t} - e^{-4t} \\ -12e^{-3t} + 12e^{-4t} & -3e^{-3t} + 4e^{-4t} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \delta(t) \quad t \geq 0$$

$$= 24e^{-3t} - 32e^{-4t} + \delta(t) \quad t \geq 0$$

$$2.34 \quad v'(t) = A v(t) + b x(t)$$

$$y(t) = c v(t) + d x(t)$$

assume $z(t) = p v(t)$ and p^{-1} exists then

$$v(t) = p^{-1} z(t) \quad v'(t) = p^{-1} z'(t) \text{ substitute into state equations}$$

$$\Rightarrow p^{-1} z'(t) = A p^{-1} z(t) + b x(t)$$

$$y(t) = c p^{-1} z(t) + d x(t)$$

$$\text{or } z'(t) = p A p^{-1} z(t) + p b x(t) = A_1 z(t) + b_1 x(t)$$

$$y(t) = c p^{-1} z(t) + d x(t) = c_1 z(t) + d_1 x(t)$$

Q.E.D

2.35

$$(a) \quad \underline{v}' = \begin{bmatrix} -3 & 1 \\ -2 & 0 \end{bmatrix} \underline{v} + \begin{bmatrix} 1 \\ -1 \end{bmatrix} x$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \underline{v}$$

$$(b) \quad \underline{z}' = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \underline{z} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} x$$

$$y = \begin{bmatrix} -1 & 1 \end{bmatrix} \underline{z}$$

$$(c) \text{ Let } P = \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix}$$

$$P A_1 = A_1 P \text{ gives } \begin{aligned} p_{11} &= p_{22} \\ 3p_{11} + 2p_{12} + p_{21} &= 0 \end{aligned}$$

$$\underline{b}_1 = P \underline{b} \text{ gives } \begin{aligned} p_{11} - p_{12} &= 0 \\ p_{21} - p_{22} &= 1 \end{aligned}$$

$$(d) \quad P = \begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix} \therefore P^{-1} = \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix}$$

$$\begin{aligned} \hat{A} &= P A_1 P^{-1} = \begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix} \\ &= \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} \end{aligned}$$

$$\hat{b} = P b_1 = \begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\hat{c} = c_1 P^{-1} = \begin{bmatrix} -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix} = \begin{bmatrix} -2 & -3 \end{bmatrix}$$

$$\text{The state equations } \begin{aligned} \hat{\underline{v}}' &= \hat{A} \hat{\underline{v}} + \hat{\underline{b}} x \\ \hat{y} &= \hat{c} \end{aligned}$$

are in diagonal form.

Chapter 3

$$3.1 \quad x_1(t) = x_{11} p_1(t) + x_{12} p_2(t)$$

$$\begin{aligned} x_{11} &= \int x_1(t) p_1(t) dt = \int_0^1 (2t) \left(\sqrt{\frac{3}{2}} t \right) dt \\ &= 2 \sqrt{\frac{3}{2}} \left. \frac{t^3}{3} \right|_0^1 \\ &= \frac{2}{3} \sqrt{\frac{3}{2}} = \sqrt{\frac{2}{3}} \end{aligned}$$

$$x_{12} = \int x_1(t) p_2(t) dt = \sqrt{\frac{2}{3}}$$

$$x_1 = \left(\sqrt{\frac{2}{3}}, \sqrt{\frac{2}{3}} \right)$$

$$x_2(t) = x_{21} p_1(t) + x_{22} p_2(t)$$

$$\begin{aligned} x_{21} &= \int x_2(t) p_1(t) dt = \int_1^2 2(t-2) \sqrt{\frac{3}{2}} (t-2) dt \\ &= 2 \sqrt{\frac{3}{2}} \int_1^2 (t-2)^2 dt \\ &= \sqrt{\frac{2}{3}} \end{aligned}$$

$$\begin{aligned} x_{22} &= \int x_2(t) p_2(t) dt = \int_1^2 2(t-2) \sqrt{\frac{3}{2}} (2-t) dt \\ &= -2 \sqrt{\frac{3}{2}} \int_1^2 (t-2)^2 dt \\ &= -\sqrt{\frac{2}{3}} \end{aligned}$$

$$\therefore x_2 = \left(\sqrt{\frac{2}{3}}, -\sqrt{\frac{2}{3}} \right)$$

$$x_3(t) = x_{31} f_1(t) + x_{32} f_2(t)$$

$$\begin{aligned} x_{31} &= \int x_3(t) f_1(t) dt = \int_0^1 3t \sqrt{\frac{3}{2}} t dt + \int_1^2 -2(t-2) \sqrt{\frac{3}{2}} (t-2) dt \\ &= 3\sqrt{\frac{3}{2}} \int_0^1 t^2 dt - 2\sqrt{\frac{3}{2}} \int_1^2 (t-2)^2 dt \\ &= 3\sqrt{\frac{3}{2}} \left(\frac{1}{3}\right) - 2\sqrt{\frac{3}{2}} \left(\frac{(t-2)^3}{3}\right) \Big|_1^2 \\ &= \sqrt{\frac{3}{2}} - \sqrt{\frac{2}{3}} = \sqrt{\frac{1}{6}} \end{aligned}$$

$$\begin{aligned} x_{32} &= \int x_3(t) f_2(t) dt = \int_0^1 3t \sqrt{\frac{3}{2}} t dt + \int_1^2 -2(t-2) \cdot \left(-\sqrt{\frac{3}{2}} (t-2)\right) dt \\ &= 3\sqrt{\frac{3}{2}} \int_0^1 t^2 dt + 2\sqrt{\frac{3}{2}} \int_1^2 (t-2)^2 dt \\ &= 3\sqrt{\frac{3}{2}} \left(\frac{1}{3}\right) + 2\sqrt{\frac{3}{2}} \left(\frac{(t-2)^3}{3}\right) \Big|_1^2 = \sqrt{\frac{9}{2}} + \sqrt{\frac{2}{3}} = \sqrt{\frac{5}{6}} \end{aligned}$$

$$\therefore x_3 = \left(\frac{1}{\sqrt{6}}, \frac{5}{\sqrt{6}} \right)$$

3.2 (a) Choose a_1 such that $\langle \psi_1, \psi_2 \rangle = 0$. Thus

$$\langle x_1, x_2 + a_1 x_1 \rangle = 0 \quad \text{or} \quad a_1 = -\frac{\langle x_1, x_2 \rangle}{E_1}$$

Choose b_1, b_2 such that $\langle \psi_1, \psi_2 \rangle = \langle \psi_2, \psi_3 \rangle = 0$. Thus

$$\langle x_1, x_3 + b_1 x_1 + b_2(x_2 + a_1 x_1) \rangle = \langle x_1, x_3 \rangle + b_1 E_1$$

so that

$$b_1 = -\frac{\langle x_1, x_3 \rangle}{E_1}$$

Also

$$\begin{aligned} \langle x_2 + a_1 x_1, x_3 + b_1 x_1 + b_2(x_2 + a_1 x_1) \rangle &= \langle x_2, x_3 \rangle + a_1 \langle x_1, x_3 \rangle \\ &\quad + b_2 \left[E_2 - \frac{\langle x_1, x_2 \rangle^2}{E_1} \right] \end{aligned}$$

so that

$$b_2 = - \frac{\langle x_1, x_2 \rangle \langle x_1, x_3 \rangle - E_1 \langle x_2, x_3 \rangle}{\langle x_1, x_2 \rangle^2 - E_1 E_2}$$

$$(b) \psi_1 = x_1, \quad E_1 = \langle \psi_1, \psi_1 \rangle = \frac{8}{3}$$

$$\langle x_1, x_2 \rangle = 0, \text{ so that } \psi_2 = x_2, \text{ and } \langle \psi_2, \psi_2 \rangle = E_2 = 2$$

$$\langle x_1, x_3 \rangle = \sqrt{\frac{3}{2}} \left[\int_0^1 (1-t) t dt + \int_1^2 (-t^2 + 3t - 2) dt \right] = \frac{1}{\sqrt{6}}$$

$$\langle x_2, x_3 \rangle = 0, \text{ so that } b_1 = -\frac{1}{\sqrt{2}\sqrt{3}} \times \frac{3}{8} = -\frac{\sqrt{3}}{8\sqrt{2}}, \quad b_2 = 0$$

Therefore

$$\psi_3 = x_3 - \frac{\sqrt{3}}{8\sqrt{2}} x_1 = \begin{cases} \sqrt{\frac{3}{2}} \left(\frac{9}{8}t - \frac{1}{8} \right) & 0 \leq t \leq 1 \\ \sqrt{\frac{3}{2}} \left(\frac{9}{8}t - \frac{17}{8} \right) & 1 \leq t \leq 2 \end{cases}$$

and

$$\langle \psi_3, \psi_3 \rangle = \frac{3}{2} \times \frac{1}{64} \left[\int_0^1 (9t-1)^2 dt + \int_1^2 (9t-17)^2 dt \right] = \frac{57}{64}$$

We therefore have

$$\phi_1(t) = \sqrt{\frac{3}{8}} \psi_1 = \sqrt{\frac{3}{8}} (1-t) \quad 0 \leq t \leq 2$$

$$\phi_2(t) = \frac{1}{\sqrt{2}} \psi_2 = \frac{1}{\sqrt{2}} \quad 0 \leq t \leq 2$$

$$\phi_3(t) = \sqrt{\frac{8}{57}} \psi_3 = \begin{cases} \frac{8}{\sqrt{38}} \left(\frac{9}{8}t - \frac{1}{8} \right) & 0 \leq t \leq 1 \\ \frac{8}{\sqrt{38}} \left(\frac{9}{8}t - \frac{17}{8} \right) & 1 \leq t \leq 2 \end{cases}$$

3.3 (a) $\psi_1 = e^{-t}u(t)$, $E_1 = \int_0^{\infty} e^{-2t} dt = \frac{1}{2}$, $E_2 = \int_0^{\infty} e^{-4t} dt = \frac{1}{4}$

$$\langle x_1, x_2 \rangle = \int_0^{\infty} e^{-3t} dt = \frac{1}{3}, \text{ so that } a_1 = -\frac{1/3}{1/2} = -\frac{2}{3}$$

and

$$\psi_2 = [e^{-2t} - \frac{2}{3}e^{-t}]u(t)$$

Now

$$\langle x_1, x_3 \rangle = \int_0^{\infty} e^{-4t} dt = \frac{1}{4}, \langle x_2, x_3 \rangle = \int_0^{\infty} e^{-5t} dt = \frac{1}{5}$$

so that

$$b_1 = -\frac{1/4}{1/2} = -\frac{1}{2}, \quad b_2 = -\frac{\frac{1}{3} \times \frac{1}{4} - \frac{1}{2} \times \frac{1}{5}}{\frac{1}{9} - \frac{1}{2} \times \frac{1}{4}} = -\frac{6}{5}$$

and

$$\begin{aligned} \psi_3 &= [e^{-3t} - \frac{1}{2}e^{-t} - \frac{6}{5}(e^{-2t} - \frac{2}{3}e^{-t})]u(t) \\ &= [e^{-3t} - \frac{6}{5}e^{-2t} + \frac{3}{10}e^{-t}]u(t) \end{aligned}$$

$$(b) \langle \psi_1, \psi_1 \rangle = \frac{1}{2}, \quad \phi_1 = \sqrt{2}\psi_1 = \sqrt{2}e^{-t}u(t)$$

$$\langle \psi_2, \psi_2 \rangle = \frac{1}{36}, \quad \phi_2 = 6\psi_2 = 6[e^{-2t} - \frac{2}{3}e^{-t}]u(t)$$

$$\langle \psi_3, \psi_3 \rangle = \frac{1}{600}, \quad \phi_3 = 10\sqrt{6}\psi_3 = 10\sqrt{6}[e^{-3t} - \frac{6}{5}e^{-2t} + \frac{3}{10}e^{-t}]u(t)$$

$$(c) \text{ Now } c_i = \int_0^{\infty} x\phi_i dt$$

so that

$$c_1 = \frac{\sqrt{2}}{5}, \quad c_2 = 6[\frac{1}{6} - \frac{2}{3} \times \frac{1}{5}] = \frac{1}{5}, \quad c_3 = 10\sqrt{6}[\frac{1}{7} - \frac{6}{5} \times \frac{1}{6} + \frac{3}{10} \times \frac{1}{5}] = \frac{\sqrt{6}}{35}$$

$$\begin{aligned}\hat{x}(t) &= c_1 \phi_1 + c_2 \phi_2 + c_3 \phi_3 \\ &= \left[\frac{4}{35} e^{-t} - \frac{6}{7} e^{-2t} + \frac{12}{7} e^{-3t} \right] u(t)\end{aligned}$$

and

$$E_x = \text{Energy in } x(t) = 0.125$$

$$E_{\hat{x}} = \text{Energy in } \hat{x}(t) = 0.1249$$

$$E_e = \text{Energy in } x(t) - \hat{x}(t) = 0.000102$$

$$3.4 \quad (a) \quad E_c(M) = \int_a^b \left| x(t) - \sum_{i=1}^M c_i \phi_i(t) \right|^2 dt$$

$$\begin{aligned}\frac{\partial E_c}{\partial c_j} &= \int_a^b \frac{\partial}{\partial c_j} \left| x(t) - \sum_{i=1}^M c_i \phi_i(t) \right|^2 dt \\ &= \int_a^b 2 \left(x(t) - \sum_{i=1}^M c_i \phi_i(t) \right) \frac{\partial}{\partial c_j} \left(x(t) - \sum_{i=1}^M c_i \phi_i(t) \right) dt \\ &= \int_a^b 2 \left(x(t) - \sum_{i=1}^M c_i \phi_i(t) \right) (-\phi_j(t)) dt \\ &= -2 \int_a^b x(t) \phi_j(t) dt + \int_a^b \sum_{i=1}^M c_i \phi_i(t) \phi_j(t) dt\end{aligned}$$

in order to minimize the energy error, the necessary condition $\frac{\partial E_c}{\partial c_j} = 0$

$$\Rightarrow c_j = \int_a^b x(t) \phi_j(t) dt.$$

(b) Consider complex functions

$$\begin{aligned} E_c(M) &= \int_a^b \left| x(t) - \sum_{i=1}^M c_i \phi_i(t) \right|^2 dt \\ &= \int_a^b \left(x(t) - \sum_{i=1}^M c_i \phi_i(t) \right) \left(x(t) - \sum_{i=1}^M c_i \phi_i(t) \right)^* dt \end{aligned}$$

$$\frac{\partial E_c(M)}{\partial c_j} = \int_a^b \left(x^*(t) - \sum_{i=1}^M c_i^* \phi_i^*(t) \right) (-\phi_j(t)) dt$$

Note that $\frac{\partial}{\partial c_j} c_j^* = 0$

Equating to zero and solving yields

$$c_j^* = \int_a^b x^*(t) \phi_j(t) dt$$

or

$$c_i = \int_a^b x(t) \phi_i^*(t) dt$$

$$3.5 \quad \int_0^T \phi_c^*(t) x(t) dt = \int_0^T \phi_c^*(t) \sum_{k=-\infty}^{\infty} c_k \phi_k(t) dt$$

$$= \sum_{k=-\infty}^{\infty} \int_0^T c_k \phi_c^*(t) \phi_k(t) dt$$

$$\int_0^T x(t) \phi_c^*(t) dt = c_i E_i$$

$$\text{or } c_i = \frac{1}{E_i} \int_0^T x(t) \phi_c^*(t) dt$$

3.6

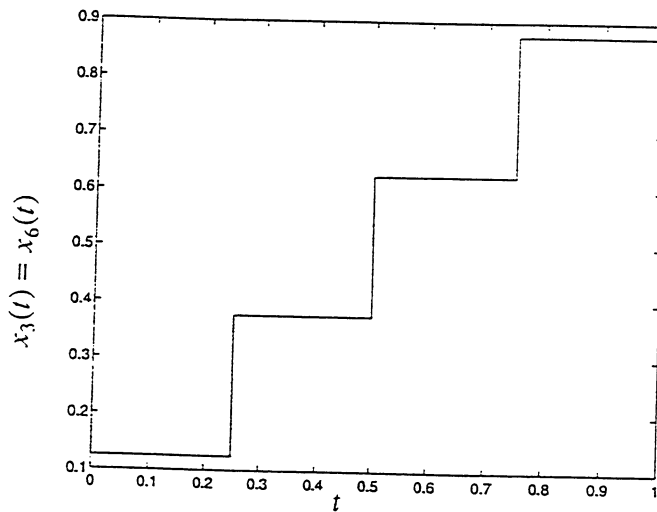
(a) We can verify that the given Walsh functions are orthonormal by directly evaluating

$$\int_0^1 \text{wal}_w(k, t) \text{wal}_w(j, t) dt, \quad 0 \leq k, j \leq 6$$

(b) Use $c_k = \int_0^1 \text{wal}_w(k, t) t dt$ to get

$$c_0 = \frac{1}{2}, c_1 = -\frac{1}{4}, c_3 = -\frac{1}{8}, c_2 = c_4 = c_5 = c_6 = 0$$

Thus $x_3(t) = x_6(t)$ and is as shown below.



3.7

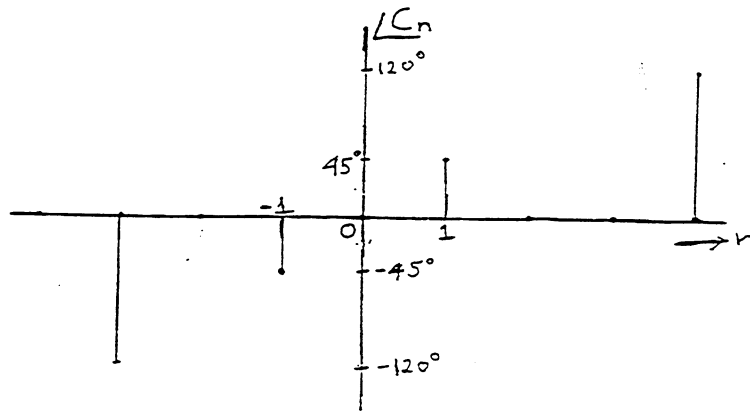
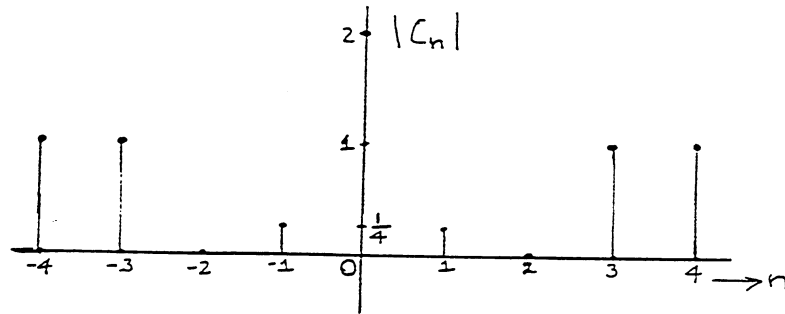
(a) With $\omega_0 = 1$, we can write

$$x(t) = 2 + \frac{1}{4} \left[e^{j(\omega_0 t + 45^\circ)} + e^{-j(\omega_0 t + 45^\circ)} \right] + \left[e^{j3\omega_0 t} + e^{-j3\omega_0 t} \right] + j \left[e^{j(4\omega_0 t + 30^\circ)} - e^{-j(4\omega_0 t + 30^\circ)} \right]$$

so that

$$c_0 = 2, c_1 = c_{-1}^* = \frac{1}{4} e^{j4^\circ}, c_3 = c_{-3}^* = 1, c_4 = c_{-4}^* = j e^{j30^\circ} = e^{j120^\circ}$$

(b)



3.8

$$x(t) = \begin{cases} 0 & -\pi < t < -\frac{\pi}{2} \\ \cos t & -\frac{\pi}{2} < t < \frac{\pi}{2} \\ 0 & \end{cases}$$

$$C_n = \frac{1}{T} \int_{-\pi}^{\pi} x(t) \exp[-jnt] dt$$

$$= \frac{1}{2\pi} \int_{-\pi/2}^{\pi/2} \cos t \exp[-jnt] dt$$

$$= \frac{1}{\pi(1-n^2)} \cos \frac{n\pi}{2} \quad |n| \neq 1$$

for $n = \pm 1$ use L'Hospital's rule

$$C_1 = C_{-1} = \frac{1}{4}$$

$$3.9 \quad x(t) = a_0 + \sum_{n=1}^{\infty} [a_n \cos(nt) + b_n \sin nt]$$

with

$$a_0 = c_0 = \frac{1}{\pi}$$

$$a_n = 2 \operatorname{Re} \{c_n\} = \frac{2}{\pi(1-n^2)} \cos \frac{n\pi}{2} \quad |n| \neq 1$$

$$b_n = 2 \operatorname{Im} \{c_n\} = 0$$

$$3.10 \quad x(t) = \sin t \quad 0 \leq t \leq \pi \quad x(t+\pi) = x(t)$$

$$\begin{aligned} c_n &= \frac{1}{\pi} \int_{\tau}^{\pi} x(t) \exp[-j2nt] dt \\ &= \frac{1}{\pi} \int_0^{\pi} \sin t \exp[-j2nt] dt = \frac{1}{\pi} \int_0^{\pi} \left(\frac{\exp[jt] - \exp[-jt]}{2j} \right) \exp[-j2nt] dt \\ &= \frac{2}{\pi} \left(\frac{1}{1-4n^2} \right) \end{aligned}$$

$$3.11 \quad x(t) = a_0 + \sum_{n=1}^{\infty} (a_n \cos 2nt + b_n \sin 2nt)$$

$$a_0 = c_0 = \frac{2}{\pi}$$

$$a_n = 2 \operatorname{Re} \{c_n\} = \frac{4}{\pi} \left(\frac{1}{1-4n^2} \right)$$

$$b_n = -2 \operatorname{Im} \{c_n\} = 0$$

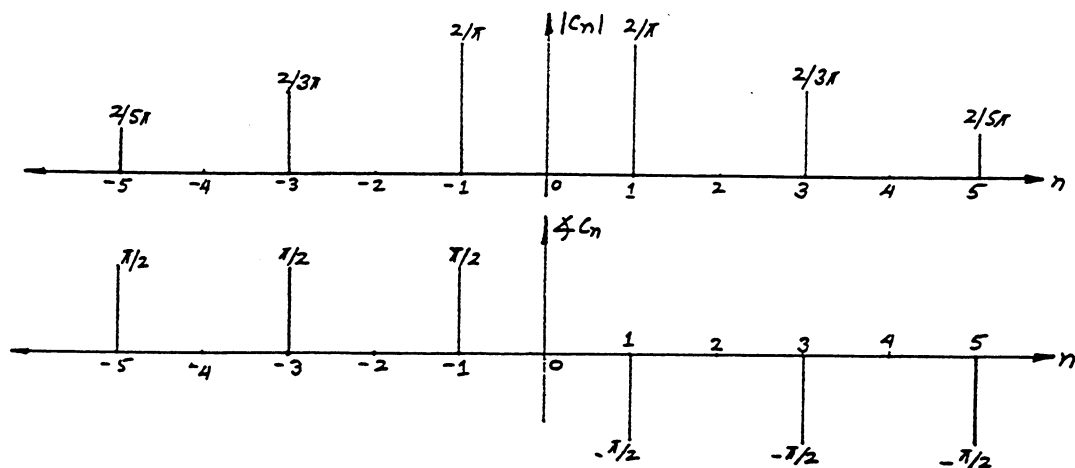
$$3.12(a) \quad \text{With } k=1 \text{ in example 3.2.1}$$

$$c_0 = 0 \quad c_n = \frac{2}{jn\pi} \quad n \text{ odd}$$

$$= 0 \quad n \text{ even}$$

$$|c_n| = \frac{2}{|n|\pi} \quad n \text{ odd}$$

$$\phi(c_n) = \begin{cases} -\pi/2 & n = 2m-1 \quad m = 1, 2, \dots \\ 0 & n = 2m \quad m = 0, 1, 2, \dots \\ \pi/2 & n = -(2m-1) \quad m = 1, 2, \dots \end{cases}$$



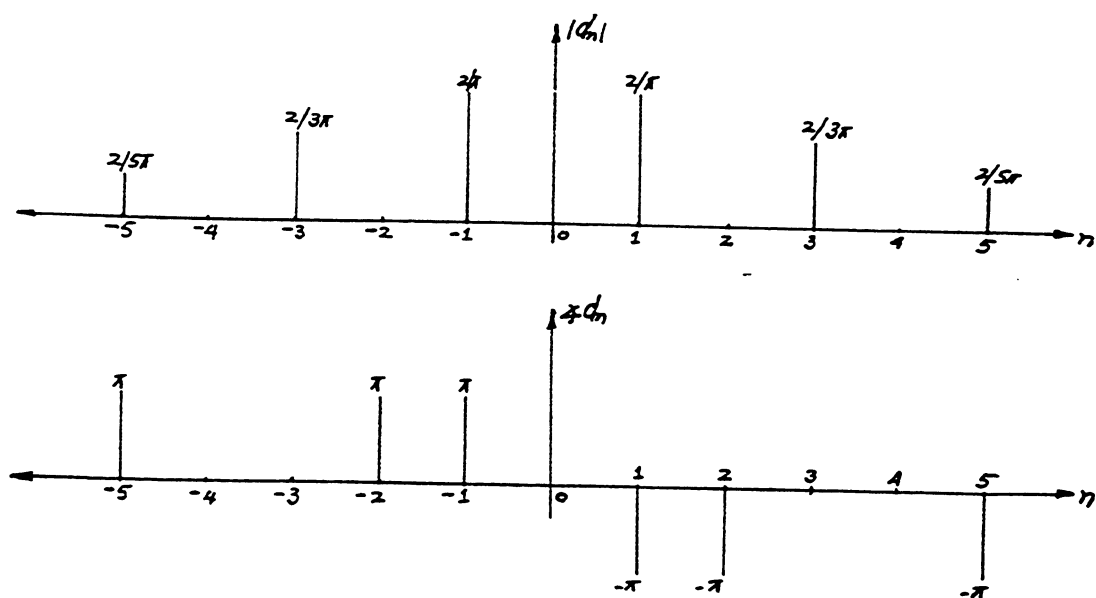
(b) The $X(t)$ in (b) is the time shift version of $X(t)$ in (a) with shift $\tau = -\frac{1}{2}$

$$d_n = c_n \exp[jn\omega_0\tau]$$

$$= c_n \exp[-jn\frac{\pi}{2}]$$

$$\Rightarrow |d_n| = |c_n|$$

$$\phi_{d_n} = \phi_{c_n} - n\frac{\pi}{2} \quad (-\pi < \phi_{d_n} < \pi)$$

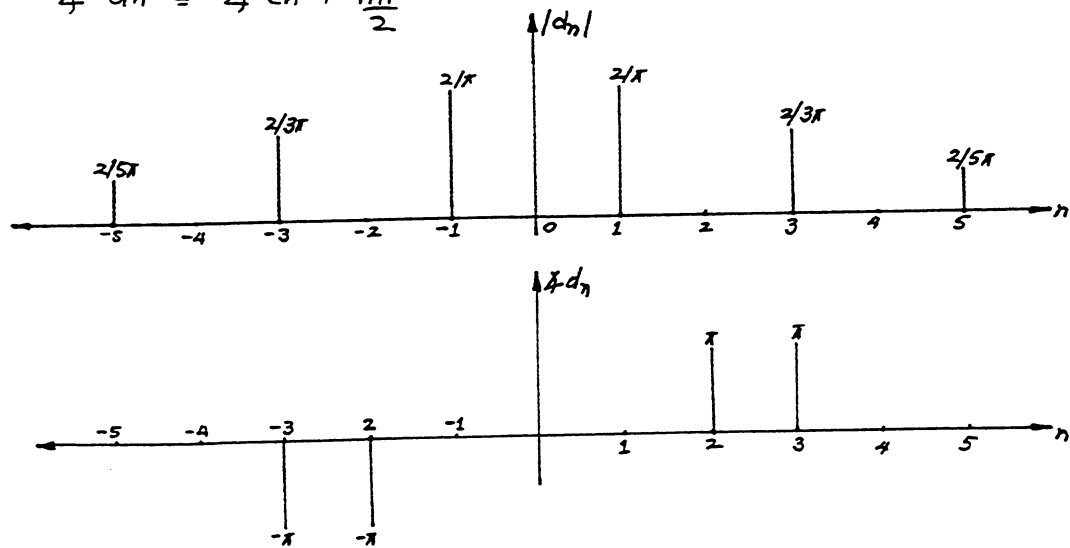


(c) The $x(t)$ in (c) is the time-shift version of $X(t)$ in (a) with shift $\tau = \frac{1}{2}$

$$d_n = C_n \exp[j n \frac{\pi}{2}]$$

$$\Rightarrow |d_n| = |C_n|$$

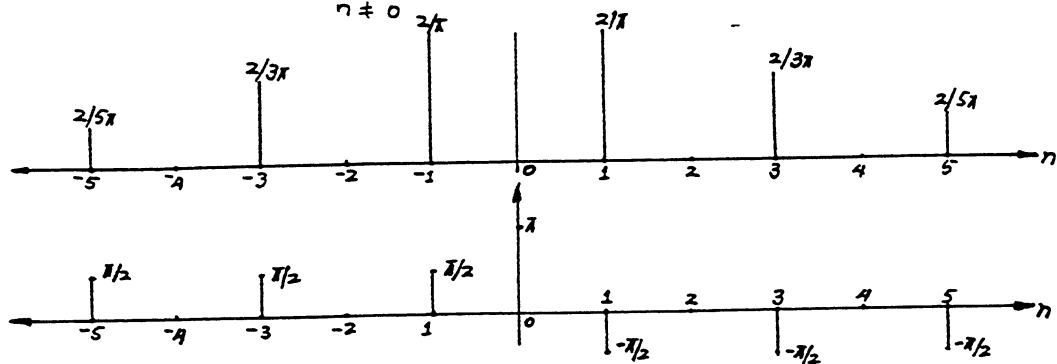
$$\angle d_n = \angle C_n + \frac{n\pi}{2}$$



(d) $X(t)$ in (d) is equal to $X(t)$ in (a) with a dc component -1

$$\text{ie } X(t) = -1 + \sum_{n=-\infty}^{\infty} C_n \exp[jn\pi t]$$

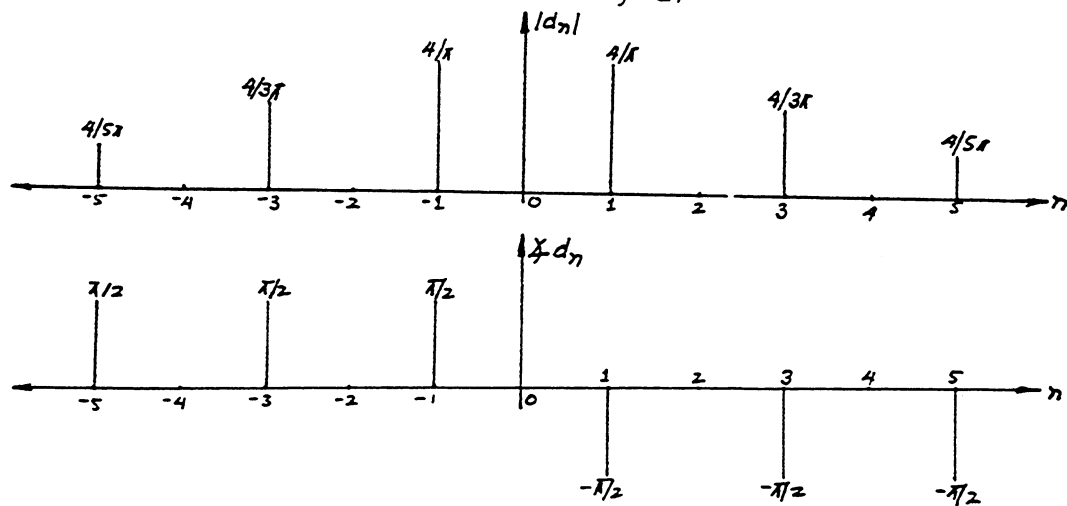
$$= -1 + \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} C_n \exp[jn\pi t]$$



(e) $X(t)$ in (e) is equal to 2 times of $X(t)$ in (a)

$$X(t) = 2 \sum_{n=-\infty}^{\infty} C_n \exp[jn\pi t]$$

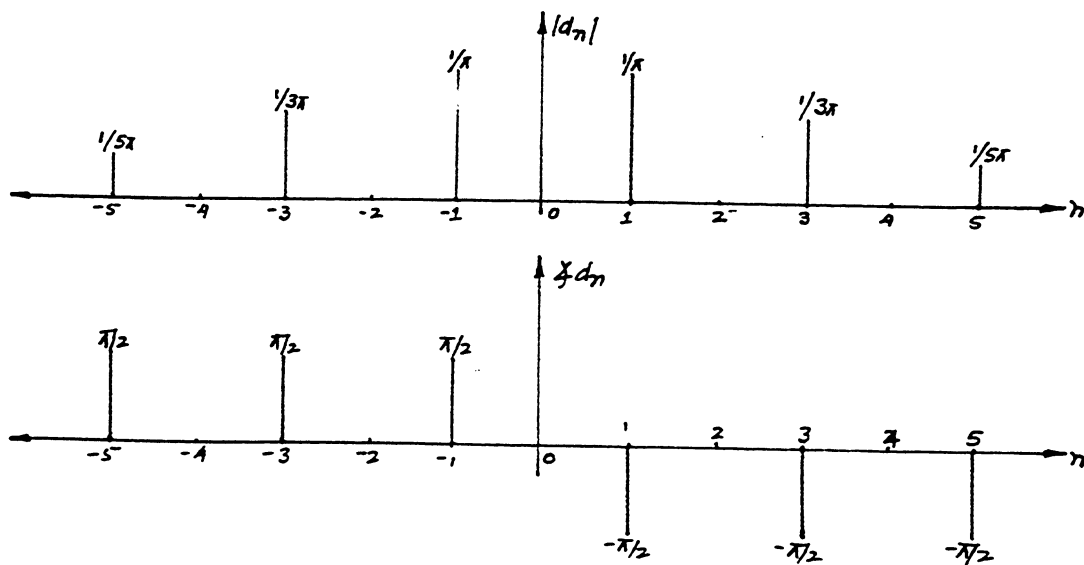
$$\Rightarrow d_n = 2C_n \quad \text{and} \quad \angle d_n = \angle C_n$$



(f) $X(t)$ in (f) is equal to one half of $X(t)$ in (a)

that is, $X(t) = \frac{1}{2} \sum_{n=-\infty}^{\infty} C_n \exp[jn\pi t]$

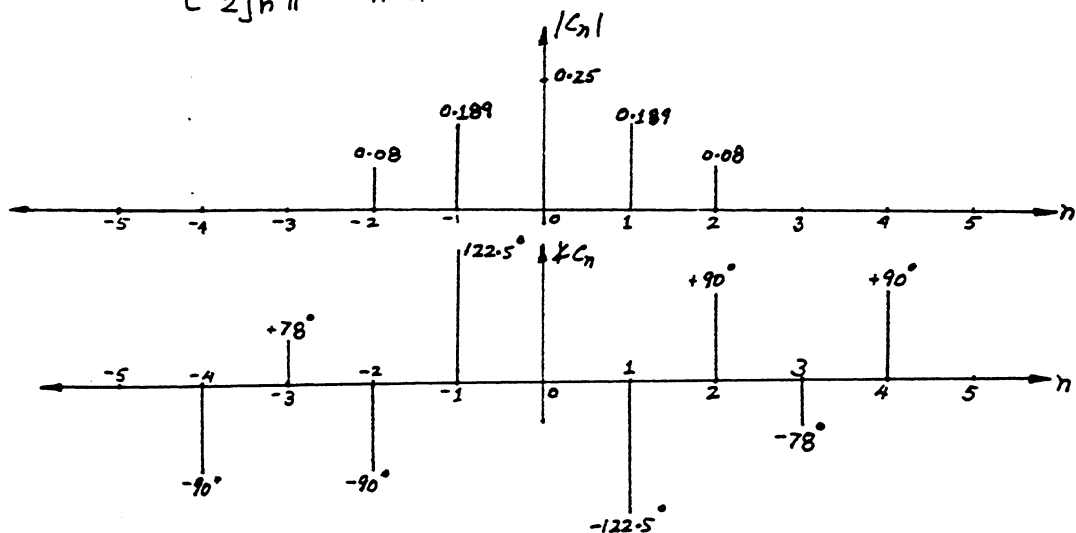
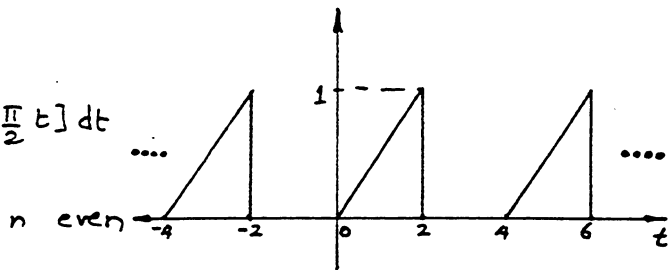
$$\Rightarrow d_n = \frac{1}{2} C_n, \quad \angle d_n = \angle C_n$$



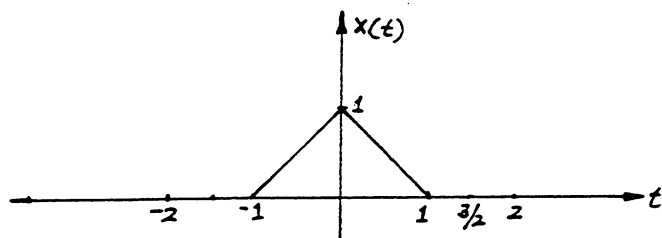
(g) $C_0 = \frac{1}{4}$

$$C_n = \frac{1}{4} \int_0^2 \frac{t}{2} \exp[-jn\frac{\pi}{2}t] dt$$

$$= \begin{cases} \frac{J}{2n\pi} \\ \frac{1}{2jn\pi} - \frac{1}{n^2\pi^2} & n \text{ odd} \end{cases}$$



(h)



$x(t)$ is an even function

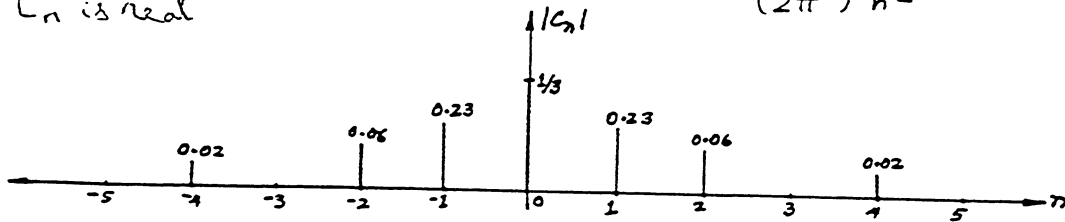
$$C_n = \frac{1}{T} \int_{-T/2}^{T/2} x(t) \exp[-jn\omega_0 t] dt$$

$$= \frac{2}{T} \int_0^{T/2} x(t) \cos n\omega_0 t dt$$

with $T=3$ $\omega_0 = \frac{2\pi}{T} = \frac{2\pi}{3}$, $C_0 = 0$

$$\begin{aligned}
 C_n &= \frac{2}{3} \int_0^1 (1-t) \cos \frac{2n\pi}{3} t \, dt \\
 &= \frac{2}{3} \left[\frac{3}{2n\pi} \sin \left(\frac{2n\pi}{3} \right) - \frac{3}{2n\pi} \left[\sin \frac{2n\pi}{3} + \frac{3}{2n\pi} \left(\cos \frac{2n\pi}{3} - 1 \right) \right] \right] \\
 &= -\frac{2}{3} \left[\left(\frac{3}{2n\pi} \right)^2 \left(\cos \frac{2n\pi}{3} - 1 \right) \right] = -\frac{3 \left[\cos \left(\frac{2n\pi}{3} \right) - 1 \right]}{(2\pi^2) n^2}
 \end{aligned}$$

C_n is real

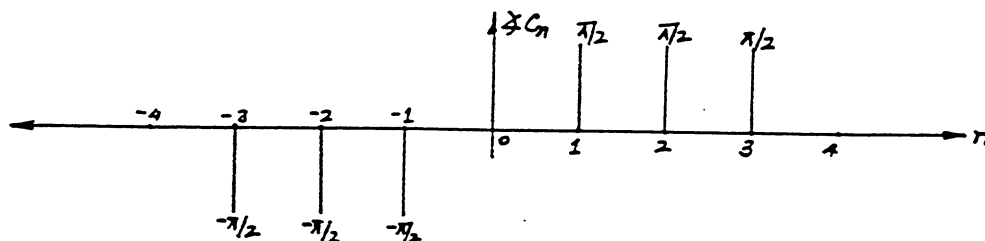
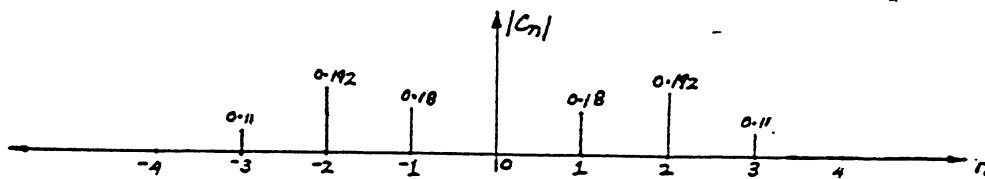


(i) $x(t)$ is an odd symmetric function

$$\begin{aligned}
 C_n &= \frac{1}{T} \int_T x(t) \exp[-jn\omega_0 t] \, dt \\
 &= -\frac{2j}{T} \int_0^{\frac{T}{2}} x(t) \sin n\omega_0 t \, dt
 \end{aligned}$$

with $T=3$ $\omega_0 = \frac{2\pi}{3}$, $C_0 = 0$

$$\begin{aligned}
 C_n &= -\frac{2j}{3} \int_0^1 (t-1) \sin \frac{2n\pi}{3} t \, dt \\
 &= -\frac{2j}{3} \left[\frac{3}{2n\pi} \left(\cos \frac{2n\pi}{3} - 1 \right) - \frac{3}{2n\pi} \cos \frac{2n\pi}{3} + \left(\frac{3}{2n\pi} \right)^2 \sin \frac{2n\pi}{3} \right] \\
 &= -\frac{2j}{3} \left[\frac{-3}{2n\pi} + \left(\frac{3}{2n\pi} \right)^2 \sin \frac{2n\pi}{3} \right] = j \left[\frac{1}{n\pi} - \frac{9}{4n^2\pi^2} \sin \frac{2n\pi}{3} \right]
 \end{aligned}$$



$$3.13 (a) \quad a_n = 2 \operatorname{Re} \{c_n\} = 0$$

$$b_n = -2 \operatorname{Im} \{c_n\} = \begin{cases} \frac{4}{n\pi} & n \text{ odd} \\ 0 & n \text{ even} \end{cases}$$

$$(b) \quad d_n = c_n \exp[-jn\frac{\pi}{2}]$$

$$\therefore a_n = 2 \operatorname{Re} \{d_n\} = \begin{cases} -\frac{4}{n\pi} \sin \frac{n\pi}{2} & n \text{ odd} \\ 0 & n \text{ even} \end{cases}$$

$$b_n = -2 \operatorname{Im} \{d_n\} = \begin{cases} -\frac{4}{n\pi} \cos \frac{n\pi}{2} & n \text{ odd} \\ 0 & n \text{ even} \end{cases}$$

$$(c) \quad d_n = c_n \exp[jn\frac{\pi}{2}]$$

$$\therefore a_n = 2 \operatorname{Re} \{d_n\} = \begin{cases} \frac{4}{n\pi} \sin \frac{n\pi}{2} & n \text{ odd} \\ 0 & n \text{ even} \end{cases}$$

$$b_n = -2 \operatorname{Im} \{d_n\} = \begin{cases} -\frac{4}{n\pi} \cos \frac{n\pi}{2} & n \text{ odd} \\ 0 & n \text{ even} \end{cases}$$

$$(d) \quad a_0 = -1, \quad a_n \text{ and } b_n \text{ are the same as in (a)}$$

$$(e) \quad d_n = 2c_n$$

$$\Rightarrow a_n = 0, \quad b_n = \begin{cases} \frac{8}{n\pi} & n \text{ odd} \\ 0 & n \text{ even} \end{cases}$$

$$(f) \quad d_n = \frac{1}{2} c_n$$

$$a_0 = 0, \quad a_n = 0, \quad b_n = \begin{cases} \frac{2}{n\pi} & n \text{ odd} \\ 0 & n \text{ even} \end{cases}$$

$$(g) \quad a_0 = c_0 = \frac{1}{4}$$

$$a_n = 2 \operatorname{Re} \{c_n\} = \begin{cases} 0 & n \text{ even} \\ -\frac{1}{n\pi} & n \text{ odd} \end{cases}$$

$$b_n = \begin{cases} -\frac{1}{n\pi} & n \text{ even} \\ \frac{1}{n\pi} & n \text{ odd} \end{cases}$$

$$(h) \quad a_0 = \frac{1}{3}, \quad a_n = -6 \frac{[\cos \frac{2n\pi}{3} - 1]}{4\pi^2 n^2}, \quad b_n = 0$$

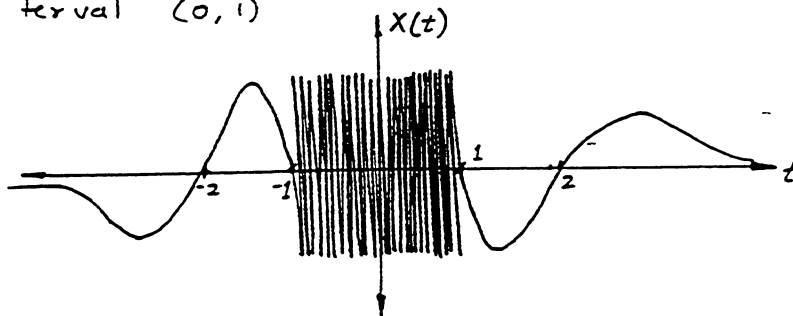
$$(i) \quad a_n = 0$$

$$b_n = -\frac{2}{n\pi} + \frac{9}{2n^2\pi^2} \sin \frac{2n\pi}{3}$$

$$3.14(a) \quad c_n = \frac{1}{T} \int_0^T x(t) \exp[-j \frac{2n\pi t}{T}] dt$$

$$|c_n| = \left| \frac{1}{T} \int_0^T x(t) \exp[-j \frac{2n\pi t}{T}] dt \right| \leq \frac{1}{T} \int_0^T |x(t)| dt < \infty$$

(b) No it does not. The function $x(t) = \sin \frac{2\pi}{t}$ does not meet Dirichlet condition 2. As illustrated in the figure below the function has an infinite number of maxima and minima in the interval $(0, 1)$.



(c) No it does not. The function $x(t) = \tan 2\pi t$ does not meet Dirichlet 1, since $\int_0^{1/2} |x(t)| dt \rightarrow \infty$

$$3.15 \quad X(t) = t^2 \quad -\pi < t \leq \pi \quad X(t+2\pi) = X(t)$$

$$(a) \quad X(t) = X(-t) \Rightarrow b_n = 0$$

$$\begin{aligned} a_n &= \frac{2}{T} \int_{-\pi}^{\pi} X(t) \cos\left(\frac{2n\pi t}{T}\right) dt \\ &= \frac{1}{\pi} \int_{-\pi}^{\pi} t^2 \cos nt \, dt = \frac{2}{\pi} \int_0^{\pi} t^2 \cos nt \, dt \\ &= \frac{2}{\pi} \left[\frac{2\pi \cos n\pi}{n^2} \right] = 4 \frac{\cos n\pi}{n^2} \end{aligned}$$

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} t^2 \, dt = \frac{1}{6\pi} t^3 \Big|_{-\pi}^{\pi} = \frac{\pi^2}{3}$$

$$(b) \quad \text{Set } t=0 \quad X(0)=0$$

$$0 = \frac{\pi^2}{3} - 4 \left(1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \dots \right)$$

$$\text{or } \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} = \frac{\pi^2}{12}$$

$$3.16 \quad C_n = \begin{cases} 0 & n=0 \\ 1 + \exp[-jn\frac{\pi}{3}] - 2 \exp[-jn\frac{\pi}{6}] & n \neq 0 \end{cases}$$

Yes it does because

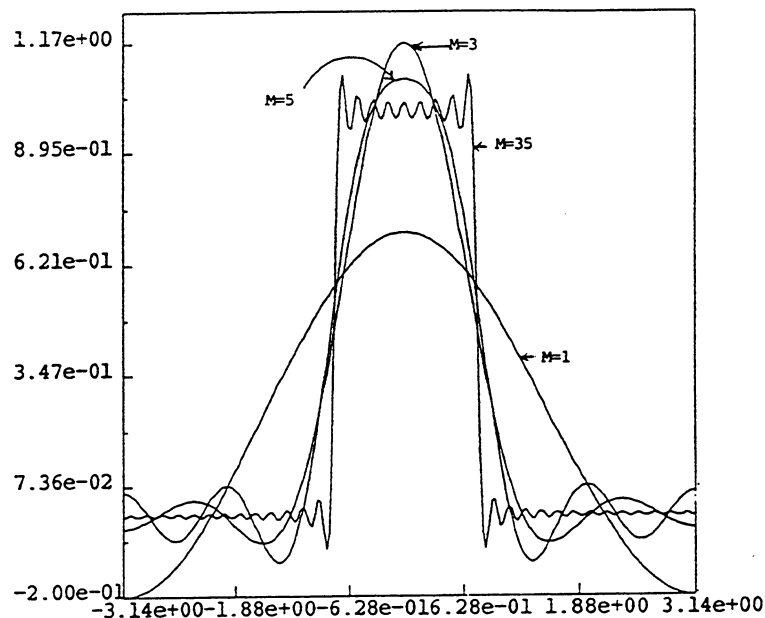
$$C_{-n} = 1 + \exp[jn\frac{\pi}{3}] - 2 \exp[jn\frac{\pi}{6}] = C_n^*$$

$$X(t) = \delta(t) + \delta(t - \frac{T}{6}) - 2\delta(t - \frac{T}{3})$$

$$3.17 \quad X(t) = \frac{1}{4} + \sum_{n=1}^M \frac{2}{n\pi} \sin \frac{n\pi}{4} \cos 2n\pi t$$

$$M=1 \quad X(t) = \frac{1}{4} + \frac{\sqrt{2}}{\pi} \cos 2\pi t$$

(a)



(b) As $M \rightarrow \infty$, we predict that

$$x(t) = 1.00 \quad |t| < \frac{\pi}{4}$$

$$= 0 \quad \frac{\pi}{4} < |t| < \pi$$

3.18 (i) $C_n = \frac{1}{T} \int_T x(t) \exp[-j \frac{2\pi n t}{T}] dt$

$$= \int_{-1/2}^{1/2} \delta(t) \exp[-j 2\pi n t] dt = 1$$

$$\therefore x(t) = \delta(t) = \sum_{n=-\infty}^{\infty} \exp[j 2\pi n t]$$

(ii) $x(t) = -\delta(t) + \delta(t-1) \quad x(t+2) = x(t)$

$$C_n = -\frac{1}{T} \int_{\langle T \rangle} x(t) \exp[-j \frac{2\pi n t}{T}] dt$$

$$= \frac{1}{2} \int_{-1/2}^{3/2} [-\delta(t) + \delta(t-1)] \exp[-j n \pi t] dt$$

$$= \frac{1}{2} [-1 + \exp[-j n \pi]] \quad \text{Thus,}$$

$$x(t) = \sum_{n \text{ odd}} \exp[j n \pi] \exp[j n \pi t]$$

$$= \begin{cases} -1 & n \text{ odd} \end{cases}$$

3.19

Suppose we consider a periodic waveform to have period NT . Then $\omega_0 = 2\pi/NT$ with Fourier coefficients

$$\begin{aligned}\hat{c}_n &= \frac{1}{NT} \int_0^{NT} x(t) e^{-jn\omega_0 t} dt \\ &= \frac{1}{NT} \left[\int_0^T x(t) e^{-jn\omega_0 t} dt + \int_T^{2T} x(t) e^{-jn\omega_0 t} dt + \dots + \int_{(N-1)T}^{NT} x(t) e^{-jn\omega_0 t} dt \right] \\ &= \frac{1}{NT} \sum_{k=0}^{N-1} \int_{kT}^{(k+1)T} x(t) e^{-jn\omega_0 t} dt\end{aligned}$$

Let $\tau = t - kT$. Then

$$\begin{aligned}\hat{c}_n &= \frac{1}{NT} \sum_{k=0}^{N-1} \int_0^T x(\tau + kT) e^{-jn\omega_0(\tau + kT)} d\tau \\ &= \frac{1}{N} \sum_{k=0}^{N-1} e^{-jn\frac{2\pi}{NT}kT} \left[\frac{1}{T} \int_0^T x(\tau) e^{-jn\omega_0\tau} d\tau \right] \\ &= \left[\frac{1}{N} \sum_{k=0}^{N-1} e^{-j\frac{2\pi}{N}kn} \right] c_n\end{aligned}$$

where c_n are the Fourier coefficients of $x(t)$ considered as periodic with period T . Now

$$\frac{1}{N} \sum_{k=0}^{N-1} e^{-j\frac{2\pi}{N}kn} = \begin{cases} 1 & \text{if } n = pN, p \text{ an integer} \\ 0 & \text{otherwise} \end{cases}$$

so that

$$\hat{c}_{pN} = c_n \quad \text{for } p = 0, \pm 1, \pm 2, \text{ etc, all other } \hat{c}_n = 0.$$

$$\text{For signal (i), } \hat{c}_{3p} = \begin{cases} 1 & p = 0, \pm 1, \pm 2, \text{ etc} \\ 0 & \text{otherwise} \end{cases}$$

$$\text{For signal (ii), } \hat{c}_{3p} = \begin{cases} -1 & p = 1, \pm 3, \pm 5, \text{ etc} \\ 0 & \text{otherwise} \end{cases}$$

3.20 (a) $c_5 = \left[\frac{\sin \frac{n\pi}{T}}{n\pi} \right]^2 = \frac{1}{T^2} = 150$, so that $T = \sqrt{150}$

(b) To find the average power in $x(t)$, we note from Equation (3.5.15) that we can consider $x(t)$ as the periodic convolution of the periodic square wave of Example 3.3.3 with itself, so that, with $K = 1$, $\tau = 1$, over the interval $[-T/2, T/2]$, we have

$$x(t) = \begin{cases} 1+t & -\frac{1}{2} \leq t \leq 0 \\ 1-t & 0 \leq t \leq \frac{1}{2} \end{cases}$$

Average power in $x(t)$ is $P = \frac{7}{12T} = 0.0476$

Average power in $\hat{x}(t)$ is $P = \sum_{n=-2}^2 |c_n|^2 = 0.1592/T = 0.0130$

- 3-21 (a) even symmetry
(b) odd symmetry
(c) half-wave odd symmetry
(d) odd symmetry
(e) half-wave even symmetry

$$x_2(t) = y(t) + z(t)$$

where $y(t) = z_1(t)$ with $\tau = 2$, $T = 4$

$z(t) = z_1(t)$ with $\tau = 3$, $T = 4$

3-22
$$Z(t) = \frac{1}{T} \int_{\langle T \rangle} x(\tau) y(t-\tau) d\tau$$

(a)
$$\begin{aligned} Z(t+T) &= \frac{1}{T} \int_{\langle T \rangle} x(\tau) y(t+T-\tau) d\tau \\ &= \frac{1}{T} \int_{\langle T \rangle} x(\tau) y(t-\tau) d\tau \end{aligned}$$

$Z(t)$ is periodic with period T

$$(b) \quad \frac{1}{T} \int_T x(\tau) y(t-\tau) d\tau = \frac{1}{T} \int_T x(t-\tau) y(\tau) d\tau \quad (\text{commutativity})$$

$$x(t) \otimes [y(t) \otimes v(t)] = \frac{1}{T} \int_{\langle \tau \rangle} x(\tau) \left[\frac{1}{T} \int_{\langle \tau \rangle} y(\sigma) v(t-\tau-\sigma) d\sigma \right] d\tau$$

$$= \frac{1}{T^2} \int_{\langle \tau \rangle} \int_{\langle \tau \rangle} x(\tau) y(\sigma) v(t-\tau-\sigma) d\sigma d\tau \quad (1)$$

$$[x(t) \otimes y(t)] \otimes v(t) = \frac{1}{T} \int_{\langle \tau \rangle} \left[\frac{1}{T} \int_{\langle \tau \rangle} x(\tau) y(\tau-\tau) d\tau \right] v(t-\tau) d\tau$$

$$= \frac{1}{T^2} \int_{\langle \tau \rangle} \int_{\langle \tau \rangle} x(\tau) y(\tau-\tau) v(t-\tau) d\tau \quad (2)$$

Let $\tau - \tau = \sigma$ in (2), yields

$$[x(t) \otimes y(t)] \otimes v(t) = \frac{1}{T^2} \int_{\langle \tau \rangle} \int_{\langle \tau \rangle} x(\tau) y(\sigma) v(t-\tau-\sigma) d\sigma d\tau \quad (3)$$

$$= x(t) \otimes [y(t) \otimes v(t)] \quad (\text{associativity})$$

3.23 The periodic convolution of $x(t)$ and $y(t)$ gives

$$z(t) = \begin{cases} \int_0^{t+\frac{1}{2}} (1-\tau) d\tau = -\frac{t^2}{2} + \frac{t}{2} + \frac{3}{8} & -\frac{1}{2} \leq t \leq \frac{1}{2} \\ \int_{t-\frac{1}{2}}^1 (1-\tau) d\tau = \frac{t^2}{2} - \frac{3t}{2} + \frac{9}{8} & \frac{1}{2} \leq t \leq \frac{3}{2} \end{cases}$$

With $\omega_0 = \pi$, the Fourier coefficients of $z(t)$ are

$$\begin{aligned} \alpha_n &= \frac{1}{2} \left[\int_{-\frac{1}{2}}^{\frac{1}{2}} \left[-\frac{t^2}{2} + \frac{t}{2} + \frac{3}{8} \right] e^{-jn\pi t} dt + \int_{\frac{1}{2}}^{\frac{3}{2}} \left[\frac{t^2}{2} - \frac{3t}{2} + \frac{9}{8} \right] e^{-jn\pi t} dt \right] \\ &= \frac{1}{2} \left[\int_{-\frac{1}{2}}^{\frac{1}{2}} [(1-e^{-jn\pi}) \left(\frac{t}{2} - \frac{t^2}{2} \right) + \left(\frac{3}{8} + \frac{1}{8} e^{-jn\pi} \right) dt] e^{-jn\pi t} \right] \end{aligned}$$

For $n = 0$, we get $\alpha_0 = \frac{1}{2} \int_{-\frac{1}{2}}^{\frac{1}{2}} (\frac{3}{8} + \frac{1}{8}) dt = \frac{1}{4}$

For $n \neq 0$, n even, $\alpha_n = \frac{1}{8} \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{-jn\pi t} dt = 0$

For n odd, $\alpha_n = \frac{1}{2} \int_{-\frac{1}{2}}^{\frac{1}{2}} [t^2 + t + \frac{1}{4}] e^{-jn\pi t} dt$

which evaluates to

$$\alpha_n = \frac{2 - jn\pi}{2n^3\pi^3} \sin \frac{n\pi}{2}$$

From Example 3.3.3, the coefficients of $x(t)$ are

$$\beta_n = \frac{\sin \frac{n\pi}{2}}{n\pi}$$

We find the coefficients for $y(t)$ as

$$\gamma_n = \frac{1}{2} \int_0^1 (1-t) e^{-jn\pi t} dt$$

$$= \begin{cases} \frac{1}{4} & n = 0 \\ \frac{-j}{2n\pi} & n \text{ even, } n \neq 0 \\ \frac{1}{n^2\pi^2} - \frac{j}{2n\pi} & n \text{ odd} \end{cases}$$

It easily follows from the above set of equations that $\alpha_n = \beta_n \gamma_n$

3.24

$$(a) \quad x(t) = \sum_{n=-\infty}^{\infty} C_n \exp[jn\omega_0 t], \quad C_0 = 0$$

$$\begin{aligned} y(t) &= \int x(t) dt = \sum_{n=-\infty}^{\infty} C_n \int \exp[jn\omega_0 t] dt \\ &= \sum_{n=-\infty}^{\infty} \frac{C_n}{jn\omega_0} \exp[jn\omega_0 t] \end{aligned}$$

$$\begin{aligned} y(t+T) &= \sum_{n=-\infty}^{\infty} \frac{C_n}{jn\omega_0} \exp[jn\omega_0(t+T)] \\ &= \sum_{n=-\infty}^{\infty} \frac{C_n}{jn\omega_0} \exp[jn\omega_0 t] = y(t) \end{aligned}$$

(b) $d_n = \frac{C_n}{jn\omega_0}$ are smaller than the amplitudes of $x(t)$

(c) Integration deemphasize the high frequency components.

(d) The integrated waveform is smoother than the original waveform.

3.25

The signal in Figure P3.14 (b) can be written in terms of the signal in part (a) as

$$\begin{aligned} x_b(t) &= 1 + x_a(2t - \pi/2) \\ &= 1 + x_a(2[t - \pi/4]) \end{aligned}$$

therefore

$$d_n = 1 + C_n \exp[-jn\frac{\pi}{2}]$$

hence

$$x_a(t) = 1 - \frac{8}{\pi^2} \left(\cos 2t + \frac{1}{9} \cos 6t + \frac{1}{25} \cos 10t + \dots \right)$$

$$3.26 \quad C_n = \frac{1}{n^2+1} \exp\left[jn\frac{\pi}{3}\right]$$

$$(a) \quad C_n^* = \frac{1}{n^2+1} \exp\left[-jn\frac{\pi}{3}\right] = C_{-n} \Rightarrow z(t) \text{ is real signal.}$$

$$(b) \quad \text{Average value} = C_0 = 1$$

$$(c) \quad RC \frac{dy(t)}{dt} + y(t) = x(t)$$

$$\text{or} \quad \frac{dy}{dt} + y(t) = x(t)$$

$$H(\omega) = \frac{1}{j\omega+1} \quad \text{or} \quad H(n\omega_0) = \frac{1}{1+jn\omega_0}$$

$$y(t) = \sum_{n=-\infty}^{\infty} \frac{1}{(1+jn\omega_0)} \frac{\exp[jn\frac{\pi}{3}]}{n^2+1} \exp[jn\omega_0 t]$$

$$n=0, \quad d_0 = C_0 = 1$$

$$n=1 \quad d_1 = \frac{1}{2(1+j\omega_0)} \exp\left[j\frac{\pi}{3}\right]$$

$$n=2 \quad d_2 = \frac{1}{5(1+j2\omega_0)} \exp\left[j2\frac{\pi}{3}\right]$$

$$d_3 = \frac{1}{10(1+j3\omega_0)} \exp[j\pi]$$

(d) for $\omega_0 \gg 1$,

the circuit deemphasize the high-frequency components

If $y(t)$ is the voltage across the resistor instead,

$$H(\omega) = \frac{1}{1+\frac{1}{j\omega}} = \frac{j\omega}{1+j\omega}; \quad H(n\omega_0) = \frac{jn\omega_0}{1+jn\omega_0}$$

$$y(t) = \sum_{n=-\infty}^{\infty} \frac{jn\omega_0}{1+jn\omega_0} \frac{\exp[jn\frac{\pi}{3}]}{1+n^2} \exp[jn\omega_0 t]$$

$$d_1 = \frac{j\omega_0}{2(1+j\omega_0)} \exp[j\frac{\pi}{3}]$$

$$d_2 = \frac{j2\omega_0}{5(1+j2\omega_0)} \exp[j\frac{2\pi}{3}]$$

$$d_3 = \frac{j3\omega_0}{10(1+j3\omega_0)} \exp[j\pi]$$

for $n\omega_0 \gg 1$, the n^{th} harmonic can be written as

$$d_n \simeq \frac{1}{n^2+1} \exp[jn\frac{\pi}{3}] = c_n$$

i.e. the circuit passes the high frequency components without alternations.

3.27

$$H(\omega) = \frac{\frac{1}{j\omega}}{1 + \frac{1}{j\omega}} = \frac{1}{1+j\omega}$$

With $\omega_0 = 1$,

$$x(t) = 1 + \frac{3}{2} e^{j30^\circ} e^{j\omega_0 t} + \frac{3}{2} e^{-j30^\circ} e^{-j\omega_0 t} + \frac{1}{2} e^{j2\omega_0 t} + \frac{1}{2} e^{-j2\omega_0 t}, \text{ where}$$

$$\text{Therefore } c_0 = 1, c_1 = c_{-1}^* = \frac{3}{2} e^{j30^\circ}, c_2 = c_{-2} = \frac{1}{2}$$

$$H(0) = 1, H(\omega_0) = \frac{1}{1+j1} = 0.7071 e^{-j45^\circ}, H(2\omega_0) = 0.4472 e^{-j63.43^\circ}$$

and

$$d_0 = 1, d_1 = d_{-1}^* = 1.0607 e^{-j15^\circ}, d_2 = d_{-2}^* = 0.2236 e^{-j63.43^\circ}$$

which gives

$$y(t) = 1 + 2.1214 \cos(t - 15^\circ) + 0.4472 \cos(2t - 63.43^\circ)$$

3.28 System 1 has magnitude and phase distortion.

System 2 is distortionless.

System 3 has magnitude distortion.

System 4 has phase distortion.

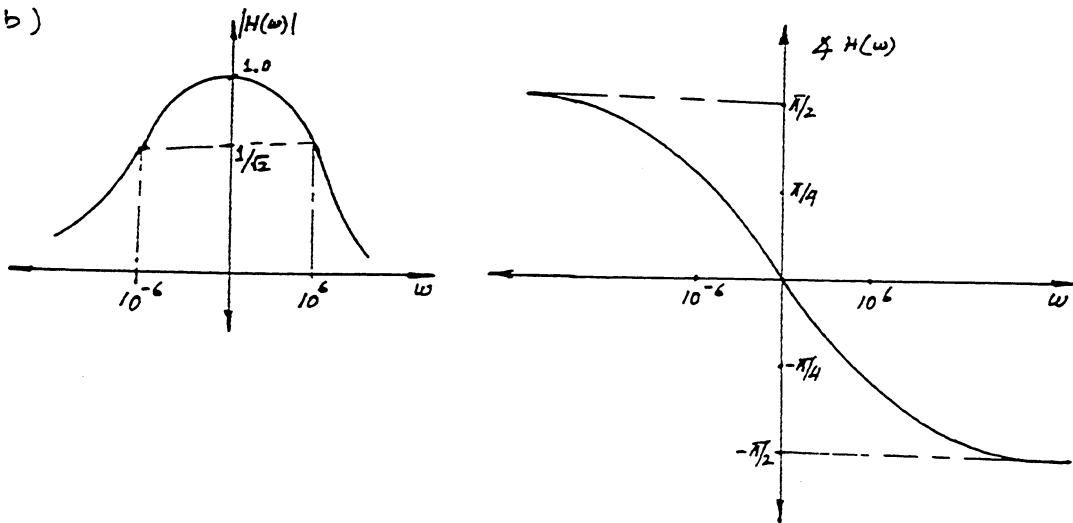
3.29 (a) For an input $x(t) = \exp[j\omega t]$, the output is

$$y(t) = \frac{\exp[j\omega t]}{R + \frac{1}{j\omega C}} \cdot \frac{1}{j\omega C}$$

$$= \frac{1}{1 + j\omega RC} \exp[j\omega t] = H(\omega) \exp[j\omega t]$$

where $H(\omega) = \frac{1}{1 + j\omega RC} = \frac{1}{1 + j10^{-6}\omega}$

(b)



(c) for $x(t) = 10 \exp[j\omega t]$

$$y(t) = 10 H(\omega) \exp[j\omega t]$$

$$\frac{|y(t) - x(t)|}{|x(t)|} = \frac{|10 H(\omega) - 10|}{10} < 0.01$$

$$\text{or } |H(\omega) - 1.0| < 0.01$$

$$\frac{|10^{-6}\omega|}{\sqrt{1 + 10^{-12}\omega^2}} < 0.01$$

$$\omega^2 < \frac{10^{-4}}{10^{-12} - 10^{-16}}$$

$$|\omega| < 10^4$$

$$(d) \quad \angle H(\omega) = -\tan^{-1} 10^{-6} \omega$$

$$\text{Need } \frac{|\tan^{-1} 10^{-6} \omega - 10^{-6} \omega|}{|10^{-6} \omega|} < 0.02$$

$$\text{which gives } \omega < 249450$$

$$3.30 \quad y(t) = A x(t) + B x^2(t)$$

$$x(t) = a_1 \cos \omega_0 t + a_2 \cos 2\omega_0 t$$

$$y(t) = A [a_1 \cos \omega_0 t + a_2 \cos 2\omega_0 t] + B [a_1 \cos \omega_0 t + a_2 \cos 2\omega_0 t]^2$$

$$= A a_1 \cos \omega_0 t + A a_2 \cos 2\omega_0 t + B [a_1^2 \cos^2 \omega_0 t + 2 a_1 a_2 \cos \omega_0 t \cos 2\omega_0 t + a_2^2 \cos^2 2\omega_0 t]$$

$$= A a_1 \cos \omega_0 t + A a_2 \cos 2\omega_0 t + B \left[a_1^2 \frac{1 + \cos 2\omega_0 t}{2} \right.$$

$$\left. + 2 a_1 a_2 \left(\frac{\cos \omega_0 t + \cos 3\omega_0 t}{2} \right) + \frac{a_2^2}{2} (1 + \cos 4\omega_0 t) \right]$$

$$= A a_1 \cos \omega_0 t + A a_2 \cos 2\omega_0 t + \frac{B a_1^2}{2} + \frac{B \cos 2\omega_0 t}{2}$$

$$+ B a_1 a_2 \cos \omega_0 t + B a_1 a_2 \cos 3\omega_0 t + \frac{B a_2^2}{2} + \frac{B a_2^2 \cos 4\omega_0 t}{2}$$

The new harmonics are $B a_1 a_2 \cos 3\omega_0 t$,

$$\frac{B a_2^2}{2} \cos 4\omega_0 t$$

Also a DC term with magnitude $\frac{-B a_1^2 + B a_2^2}{2}$ was generated.

3.31 Now $f_0 = 2$ kHz and $c_n = (K\tau/T)\text{sinc}(n\tau/T) = (1/5)\text{sinc}(n/5)$

The filter passes only the dc, 1st and 2nd harmonics.

For $0 \leq |f| \leq 5$ kHz, $|H(f)| = 10$, $\text{Arg } H(f) = -(\pi f)/10000$ rads.

Now $c_0 = \frac{1}{5}$, $c_1 = c_{-1}^* = \frac{1}{5} \text{sinc}(\frac{1}{5})$, $c_2 = c_{-2}^* = \frac{1}{5} \text{sinc}(\frac{2}{5})$

so that

$$d_0 = 2, d_1 = d_{-1}^* = 2\text{sinc}(\frac{1}{5})e^{-j\frac{\pi}{5}}, d_2 = d_{-2}^* = 2\text{sinc}(\frac{2}{5})e^{-j\frac{2\pi}{5}}$$

and

$$y(t) = 2 + \text{sinc}(\frac{1}{5})\cos(4000\pi t - 36^\circ) + \text{sinc}(\frac{2}{5})\cos(8000\pi t - 72^\circ)$$

3.32 From Example 3.4.2, $a_n = \begin{cases} \frac{8A}{n\pi} & n \text{ odd} \\ 0 & n \text{ even} \end{cases}$

With $b_n = 0$, we get $c_n = \begin{cases} \frac{4A}{n\pi} & n \text{ odd} \\ 0 & n \text{ even} \end{cases}$

For the given circuit, $H(\omega) = 1/(R + j\omega L) = 1/(100 + j0.5\pi)$

so that with $\omega_0 = \pi/2$, we have

$$d_n = \frac{1}{100 + j0.5\pi n} \times \frac{40}{n\pi} \quad n \text{ odd}$$

and

$$|d_n| = \frac{1}{(100)^2 + (0.5\pi n)^2} \times \frac{1600}{(n\pi)^2} \quad n \text{ odd}$$

The average power dissipated in the resistor is

$$P = \sum_{n \text{ odd}} |d_n|^2 \approx 0.018$$

3.33 From Example 3.3.2

$$c_n = \begin{cases} \frac{E}{\pi(1-n^2)} & n \text{ even} \\ 0 & n \text{ odd, } n \neq \pm 1 \end{cases}$$

$$c_1 = c_{-1}^* = \frac{E}{4j}$$

Since $H(\omega) = 2/(j\omega + 3)$,

$$d_0 = \frac{2E}{3\pi}$$

$$d_1 = d_{-1}^* = -j \frac{E}{4} \times \frac{2}{j\omega_0 + 3} = \frac{E}{2\sqrt{9 + \omega_0^2}} - \frac{\pi}{2} - \tan^{-1} \frac{\omega_0}{3}$$

$$d_2 = d_{-2}^* = -\frac{E}{3\pi} \times \frac{2}{j2\omega_0 + 3} = \frac{2E}{3\sqrt{9 + 4\omega_0^2}} - \pi - \tan^{-1} \frac{2\omega_0}{3}$$

3.34 With $H(\omega) = \frac{j\omega + 1}{(j\omega)^2 + 3j\omega + 2} = \frac{1}{j\omega + 2}$, we have

$$d_0 = \frac{2E}{3\pi}$$

$$d_1 = d_{-1}^* = \frac{E}{2\sqrt{4 + \omega_0^2}} \left[-\frac{\pi}{2} - \tan^{-1} \frac{\omega_0}{2} \right]$$

$$d_2 = d_{-2}^* = \frac{2E}{6\sqrt{1 + \omega_0^2}} \left[-\pi - \tan^{-1} \omega_0 \right]$$

3.35 $y_c(t) = \int_{-\infty}^{\infty} x(\tau) \cos n\omega_0 \tau h(t-\tau) d\tau = \frac{1}{T_1} \int_{t-T_1}^t x(\tau) \cos n\omega_0 \tau d\tau$

$$y_s(t) = \int_{-\infty}^{\infty} x(\tau) \sin n\omega_0 \tau h(t-\tau) d\tau = \frac{1}{T_1} \int_{t-T_1}^t x(\tau) \sin n\omega_0 \tau d\tau$$

$$\text{Also } C_n = \frac{1}{T} \int_0^T x(\tau) \exp[-jn\omega_0 \tau] d\tau$$

$$= \frac{1}{T} \int_0^T x(\tau) \cos n\omega_0 \tau d\tau - j \frac{1}{T} \int_0^T x(\tau) \sin n\omega_0 \tau d\tau$$

$$\Rightarrow y_c(t) - j y_s(t)$$

$$(a) \quad T_1 = T \quad y_c(t) = \frac{1}{T} \int_{t-T}^t x(\tau) \cos n\omega_0 \tau d\tau$$

$$= \frac{1}{T} \int_0^T x(\tau) \cos n\omega_0 \tau d\tau = \operatorname{Re} \{C_n\}$$

$$y_s(t) = \frac{1}{T} \int_{t-T}^t x(\tau) \sin n\omega_0 \tau d\tau = \frac{1}{T} \int_0^T x(\tau) \sin n\omega_0 \tau d\tau$$

$$= -\operatorname{Im} \{C_n\}$$

$$(b) \quad T_1 = 2T,$$

$$y_c(t) = \frac{1}{2T} \int_{t-2T}^t x(\tau) \cos n\omega_0 \tau d\tau = \frac{1}{2T} \int_0^T x(\tau) \cos n\omega_0 \tau d\tau$$

$$= \frac{1}{T} \int_0^T x(\tau) \cos n\omega_0 \tau d\tau$$

$$= \operatorname{Re} \{C_n\}$$

$$y_s(t) = \frac{1}{2T} \int_{t-2T}^t x(\tau) \sin n\omega_0 \tau d\tau$$

$$= \frac{1}{2T} \int_0^T x(\tau) \sin n\omega_0 \tau d\tau = \frac{1}{T} \int_0^T x(\tau) \sin n\omega_0 \tau d\tau$$

$$= -\operatorname{Im} \{C_n\}$$

$$(c) \quad T_1 \gg T \quad \text{but} \quad T_1 \neq 2T$$

$$\text{assume } T_1 = 2T + fT \quad l = \text{integer}, \quad 0 < f < 1 \quad \text{and } l \gg f$$

$$y_c(t) = \frac{1}{T_1} \int_{t-T_1}^t x(\tau) \cos n\omega_0 \tau d\tau = \frac{1}{2T+fT} \int_{t-T_1}^{(t-T_1)+2T} x(\tau) \cos n\omega_0 \tau d\tau$$

$$+ \frac{1}{2T+fT} \int_{(t-T_1)+2T}^{(t-T_1)+2T+fT} x(\tau) \cos n\omega_0 \tau d\tau$$

$$\text{Since } 2T \gg fT \quad y_c(t) \simeq \frac{1}{2T} \int_0^{2T} x(\tau) \cos n\omega_0 \tau d\tau$$

$$+ \frac{1}{2T} \int_0^{fT} x(\tau) \cos n\omega_0 \tau d\tau$$

$$\simeq \frac{1}{T} \int_0^T x(\tau) \cos n\omega_0 \tau d\tau = \operatorname{Re} \{C_n\}$$

$$\text{Similarly} \quad y_s(t) \simeq -\operatorname{Im} \{C_n\}$$

$$3.36 \quad H(\omega) = \frac{R_2}{(R_1 + R_2) + j\omega R_1 R_2 C}$$

$$= \frac{1}{2 + j5\omega \times 10^{-2}}$$

$$|d_2| = |C_2 H(2\omega_0)|, \text{ with } \omega_0 = 1$$

$$= \left| \frac{1}{\pi(1-4)} \cos \pi \frac{1}{2 + j10 \times 10^{-2}} \right| = 0.05298$$

$$|d_4| = |C_4 H(4\omega_0)|$$

$$= \left| \frac{1}{\pi(1-16)} \cos 2\pi \frac{1}{2 + j20 \times 10^{-2}} \right| = 0.01056$$

$$3.37 \quad H(\omega) = \frac{R}{(R - \omega^2 LRC) + j\omega L}$$

$$= \frac{1}{(1 - 10^{-5}\omega^2) + j10^{-4}\omega}$$

$$|d_2| = |C_2 H(2\omega_0)|, \text{ with } \omega_0 = 1$$

$$= \left| \frac{1}{\pi(1-4)} \cos \pi \frac{1}{(1-10^{-5}) + j10^{-4}} \right| = 0.1061$$

$$|d_4| = |C_4 H(4\omega_0)|$$

$$= \left| \frac{1}{\pi(1-16)} \cos 2\pi \frac{1}{(1-4 \times 10^{-5}) + j2 \times 10^{-4}} \right| = 0.02122$$

$$3.38 \quad (a) \ c_n = \begin{cases} \frac{2}{\pi(1-n^2)} & n \text{ even} \\ 0 & n \text{ odd} \end{cases}$$

For the circuit of Figure 3.36, since $H(\omega) = 20/(40 + j\omega)$

$$d_0 = \frac{1}{\pi}, \quad |d_2| = \frac{40}{3\pi\sqrt{600 + \omega_0^2}}$$

For the circuit of Figure 3.37, since

$$H(\omega) = \frac{10^4}{10^4 - 0.1\omega^2 + j\omega}$$

$$d_0 = \frac{2}{\pi}, \quad |d_2| = \frac{2 \times 10^4}{3\pi\sqrt{(10^4 - 0.1\omega^2)^2 + \omega^2}}$$

(b) We have

$$c_n = \begin{cases} \frac{4}{n\pi} & n \text{ odd} \\ 0 & n \text{ even} \end{cases}$$

For the system of Figure 3.36, we get

$$d_1 = d_{-1}^* = \frac{800}{\pi(40 + j\omega_0)}$$

For the system of Figure 3.36, we get

$$d_1 = d_{-1}^* = \frac{40 \times 10^4}{\pi(10^4 - 0.1\omega^2 + j\omega)}$$

$$\begin{aligned}
 3.39(a) \quad \sum_{n=-N}^N \exp[jn\omega_0 t] &= \frac{\exp[-jN\omega_0 t] - \exp[j(N+1)\omega_0 t]}{1 - \exp[j\omega_0 t]} \\
 &= \frac{\exp[j\frac{\omega_0 t}{2}] [\exp[-j(N+\frac{1}{2})\omega_0 t] - \exp[j(N+\frac{1}{2})\omega_0 t]]}{\exp[j\frac{\omega_0 t}{2}] [\exp[-j\frac{\omega_0 t}{2}] - \exp[j\frac{\omega_0 t}{2}]]} \cdot \frac{1}{2j} \\
 &= \frac{\sin[(N+\frac{1}{2})\omega_0 t]}{\sin \frac{\omega_0 t}{2}}
 \end{aligned}$$

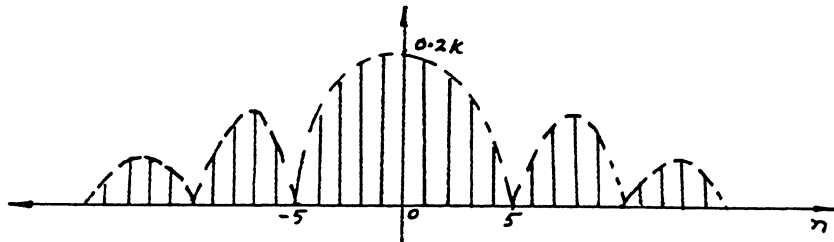
$$\begin{aligned}
 (b) \quad \frac{1}{T} \int_{-T/2}^{T/2} \frac{\sin(N+\frac{1}{2})\omega_0 t}{\sin(\omega_0 t/2)} dt, \quad \omega_0 = \frac{2\pi}{T} \\
 = \frac{1}{T} \int_{-T/2}^{T/2} \sum_{n=-N}^N \exp[jn\omega_0 t] dt = \frac{1}{T} \sum_{n=-N}^N \int_{-T/2}^{T/2} \exp[jn\omega_0 t] dt \\
 = \frac{1}{T} \sum_{n=-N}^N \left. \frac{\exp[jn\omega_0 t]}{jn\omega_0} \right|_{-T/2}^{T/2} \\
 = \sum_{n=-N}^N \left(\frac{\exp[jn\pi] - \exp[-jn\pi]}{j2\pi n} \right)
 \end{aligned}$$

note for $n \neq 0$, $\frac{\exp[jn\pi] - \exp[-jn\pi]}{j2\pi n} = 0$

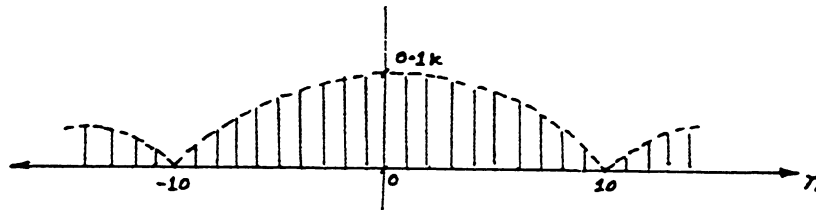
$$\begin{aligned}
 \therefore \frac{1}{T} \int_{-T/2}^{T/2} \frac{\sin(N+\frac{1}{2})\omega_0 t}{\sin(\omega_0 t/2)} dt &= \sum_{n=-N}^N \left(\frac{\exp[jn\pi] - \exp[-jn\pi]}{j2\pi n} \right) \\
 &= 1
 \end{aligned}$$

$$3.40 \quad C_n = \frac{K\tau}{T} \sin \tau (n\pi \tau / T) \quad (\text{from } 3.2.15)$$

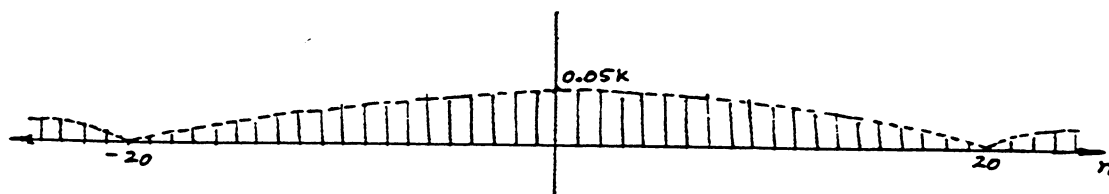
$$(i) \quad \frac{\tau}{T} = 0.2 \quad |C_n| = 0.2 K \sin \tau (0.2 n\pi)$$



$$(ii) \quad \frac{\tau}{T} = 0.1 \quad |C_n| = 0.1 K \sin \tau (0.1 n\pi)$$



$$(iii) \quad \frac{\tau}{T} = 0.05 \quad |C_n| = 0.05 K \sin \tau (0.05 n\pi)$$



Conclusions :

- (1) the magnitude of the spectrum increases proportional to τ ,
- (2) the frequency content of the signal is compressed when τ increases.

$$3.41 \quad C_n = \frac{k}{n\pi} \sin\left(\frac{n\pi r_0}{T}\right)$$

$$\tilde{C}_2 = \frac{k}{2\pi} \sin\left(\frac{2\pi r}{T}\right), \quad r = r_0 - \epsilon$$

$$= \frac{k}{2\pi} \sin\left(\frac{2\pi}{T}(r_0 - \epsilon)\right)$$

$$= \frac{k}{2\pi} \sin\left(\frac{2\pi}{T}r_0 - \frac{2\pi\epsilon}{T}\right) = \frac{k}{2\pi} \sin\frac{2\pi}{T}r_0 \cos\frac{2\pi\epsilon}{T}$$

$$- \frac{k}{2\pi} \cos\frac{2\pi}{T}r_0 \sin\frac{2\pi\epsilon}{T}$$

When $\epsilon \ll r_0$ i.e. $\epsilon \rightarrow 0$

$$\sin\frac{2\pi\epsilon}{T} \approx \frac{2\pi\epsilon}{T}, \quad \cos\frac{2\pi\epsilon}{T} \approx 1$$

$$\tilde{C}_2 \approx \frac{k}{2\pi} \sin\frac{2\pi}{T}r_0 - \frac{k}{2\pi} \frac{2\pi\epsilon}{T} \cos\frac{2\pi}{T}r_0$$

$$= \frac{k}{2\pi} \sin\frac{2\pi}{T}r_0 - \frac{k\epsilon}{T} \cos\frac{2\pi}{T}r_0$$

At $T=10$, $r_0=1$ $\epsilon=0.1$

$$C_2 \text{ exact} = \frac{k}{2\pi} \sin\left(\frac{\pi}{5}\right) = 0.0935 k$$

$$\% \text{ change} = \left| \frac{\tilde{C}_2 - C_2}{C_2} \right| \times 100$$

$$= \left| \frac{\frac{k\epsilon}{T} \cos\frac{2\pi}{T}r_0}{\frac{k}{2\pi} \sin\frac{2\pi}{T}r_0} \right| \times 100$$

$$= 100 \left| \frac{\epsilon}{2\pi} \cot\frac{2\pi}{T}r_0 \right|$$

$$= 100 \times \frac{0.1}{2\pi} \times 1.37638$$

$$= 2.19 \%$$

$$3.42 \quad (a) \quad c_n = \begin{cases} \frac{1}{\pi} \left[A \left(\frac{\sin(n-1)t_0}{n-1} - \frac{\sin(n+1)t_0}{n+1} \right) + 2B \frac{\cos nt_0}{n} \right], & n \text{ odd} \\ 0, & n \text{ even} \end{cases}$$

$$(b) \quad |c_3| = \frac{A}{12\pi} |2 \sin 2t_0 + \sin 4t_0|$$

when $B = A/2$, $t_0 = 0.5236 \text{ rad}$,

in this case

$$|c_3| = 0.068916 \text{ A}$$

(c) $|c_3|$ is maximum at $t_0 = 30^\circ$ and 150°

$$3.43 \quad (a) \quad c_n = \begin{cases} \frac{-A}{2\pi} \left[\frac{2}{n^2-1} + \frac{\exp[-j(n+1)t_0]}{n+1} - \frac{\exp[-j(n-1)t_0]}{n-1} \right], & n \text{ odd} \\ 0, & n \text{ even} \end{cases}$$

(b) $|c_3|$ is maximum at $t_0 = 2.234 \text{ rad}$
(128°)

$$|c_3|_{\max} = 0.99691 \text{ A}$$

(c) The method of generating the 3rd harmonic in problem 3.37 is more efficient.

$$3.44 \quad (a) \quad \omega t_1 = \pi - \tan^{-1}(2\pi \times 6.0)$$

$$= 1.5973$$

$$A = \sin(1.5973) \exp\left[\frac{1.5973}{37.7}\right]$$

$$= 1.0429$$

$$A \exp\left[\frac{-t_2}{Rc}\right] = 0.85895 = \sin \omega t_2$$

$$\begin{aligned}
 (b) \quad c_n &= \int_{\pi/2\omega}^{t_2} A \exp[-\sigma t] \exp[-jnt] dt \\
 &\quad + \int_{t_2}^{5\pi/2\omega} \sin t \exp[-jnt] dt \\
 &= \frac{A}{\sigma + jn} \left[\exp\left[-(\sigma + jn)\frac{\pi}{2\omega}\right] - \exp\left[-(\sigma + jn)t_2\right] \right] \\
 &\quad + \frac{1}{2(n-1)} \left[\exp\left[-jn(n-1)\frac{5\pi}{2\omega}\right] - \exp\left[-jn(n-1)t_2\right] \right] \\
 &\quad - \frac{1}{2(n+1)} \left[\exp\left[-jn(n+1)\frac{5\pi}{2\omega}\right] - \exp\left[-jn(n+1)\frac{t_2}{2}\right] \right]
 \end{aligned}$$

$$(c) \quad C_0 = 1.472 \times 10^{-2}$$

$$|C_1| = 1.3541 \times 10^{-2}$$

$$\frac{|C_1|}{C_0} = 0.92126$$

CHAPTER 4

4.1 (a) $x(-t) \longleftrightarrow x(-\omega)$

(b) $x_e(t) = (x(t) + x(-t))/2 \longleftrightarrow (X(\omega) + X(-\omega))/2$

(c) $x_o(t) = (x(t) - x(-t))/2 \longleftrightarrow (X(\omega) - X(-\omega))/2$

(d) $x^*(t) \longleftrightarrow X^*(-\omega)$

(e) $\text{Re}\{x(t)\} = (x(t) + x^*(t))/2 \longleftrightarrow (X(\omega) + X^*(-\omega))/2$

(f) $\text{Im}\{x(t)\} = (x(t) - x^*(t))/2j \longleftrightarrow (X(\omega) - X^*(-\omega))/2j$

4.2 (a) $x(t) = e^{-2t} u(-t)$ is not absolutely integrable.

$X(\omega)$ does not exist

(b) $x(t) = |t| u(t) = t u(t)$ is not absolutely integrable.

No Fourier transform

(c) $x(t) = \cos \frac{\pi}{t}$ is not well behaved at $t=0$. No $X(\omega)$.

(d) $x(t) = \frac{1}{t}$ is not absolutely integrable. No $X(\omega)$

(e) $x(t) = t^2 e^{-t} u(t)$ is absolutely integrable.

$X(\omega)$ exists.

$$\begin{aligned} 4.3 \quad X(\omega) &= \int_{-\infty}^{\infty} x(t) \exp[-j\omega t] dt \\ &= \int_{-\infty}^{\infty} x(t) \sum_{n=0}^{\infty} \frac{[-j\omega t]^n}{n!} dt \\ &= \sum_{n=0}^{\infty} \frac{(-j\omega)^n}{n!} \int_{-\infty}^{\infty} t^n x(t) dt \\ &= \sum_{n=1}^{\infty} (-j)^n m_n \frac{\omega^n}{n!} \end{aligned}$$

$$4.4 \quad \int_{-\infty}^{\infty} \cos \omega t d\omega = \lim_{\Omega \rightarrow \infty} \int_{-\Omega}^{\Omega} \cos \omega t d\omega = \lim_{\Omega \rightarrow \infty} \frac{2 \sin \omega t}{t} = 2\pi \delta(t)$$

$$4.5 \quad (a) \quad \mathcal{F}\{x(-t)\} = X(-\omega) = \text{rect}\left[\frac{\omega+1}{2}\right]$$

$$(b) \quad \mathcal{F}\{t x(t)\} = j \frac{dX(\omega)}{d\omega} = j [\delta(\omega) - \delta(\omega-2)]$$

$$(c) \quad \mathcal{F}\{x(t+1)\} = X(\omega) e^{j\omega} = e^{j\omega} \text{rect}\left[\frac{\omega-1}{2}\right]$$

$$(d) \quad \mathcal{F}\{x(-2t+4)\} = \mathcal{F}\{x[-2(t-2)]\} = \frac{1}{2} e^{j2\omega} X\left(-\frac{\omega}{2}\right) \\ = \frac{1}{2} e^{j2\omega} \text{rect}\left[\frac{\omega+2}{4}\right]$$

$$(e) \quad \mathcal{F}\{(t-1)x(t+1)\} = \mathcal{F}\{(t+1)x(t+1) - 2x(t-1)\} \\ = j e^{j\omega} \frac{dX(\omega)}{d\omega} - 2 e^{j\omega} X(\omega) \\ = j \delta(\omega) - j e^{j\omega} \delta(\omega-2) - 2 e^{j\omega} \text{rect}\left(\frac{\omega-1}{2}\right)$$

$$(f) \quad \mathcal{F}\left\{\frac{dx(t)}{dt}\right\} = j\omega X(\omega) = j\omega \text{rect}\left(\frac{\omega-1}{2}\right)$$

$$(g) \quad \mathcal{F}\left\{t \frac{dx(t)}{dt}\right\} = j \frac{d[j\omega X(\omega)]}{d\omega} \\ = -\omega \frac{dX(\omega)}{d\omega} - X(\omega) \\ = 2\delta(\omega-2) - \text{rect}\left(\frac{\omega-1}{2}\right)$$

$$(h) \quad \mathcal{F}\{x(2t-1)\} = \mathcal{F}\left\{x\left(2\left(t-\frac{1}{2}\right)\right)\right\} = \mathcal{F}\{x(2t)\} e^{-j\omega/2} \\ = \frac{1}{2} e^{-j\omega/2} \text{rect}\left[\frac{\omega-2}{4}\right]$$

$$\therefore \mathcal{F}\{x(2t-1) e^{-j2t}\} = \frac{1}{2} e^{-j\frac{(\omega+2)}{2}} \text{rect}\left[\frac{\omega}{4}\right]$$

$$(i) \quad \mathcal{F}\{x(t) e^{-j2t}\} = X(\omega+2) = \text{rect}\left(\frac{\omega+1}{2}\right)$$

$$(j) \quad \mathcal{F}\{t x(t) e^{-j2t}\} = j \frac{d\mathcal{F}\{x(t) e^{-j2t}\}}{d\omega} \\ = j \frac{d}{d\omega} \text{rect}\left(\frac{\omega+1}{2}\right) = j [\delta(\omega+1) - \delta(\omega)]$$

$$(k) \quad \mathcal{F}\{(t-1)x(t-1)e^{-j2t}\} = \mathcal{F}\{(t-1)x(t-1)e^{-j2(t-1)}e^{-j2}\} \\ = e^{-j2} e^{-j\omega} \int [\delta(\omega+2) - \delta(\omega)]$$

$$(l) \quad \mathcal{F}\left\{\int_{-\infty}^t x(\tau) d\tau\right\} = \frac{X(\omega)}{j\omega} + \pi X(0) \delta(\omega) \\ = \frac{1}{j\omega} \text{rect}\left(\frac{\omega-1}{2}\right) + \frac{\pi}{2} \delta(\omega)$$

$$4.6 \quad X(\omega) = \frac{1}{2+j\omega} \\ \therefore Y(\omega) = \frac{e^{-j\omega} + e^{j\omega}}{2+j\omega} = \frac{2\cos\omega}{2+j\omega}$$

$$4.7 \quad \text{From (4.2.14), } \int_{-\infty}^{\infty} \exp[j\omega_0 t] \exp[-j\omega t] dt = 2\pi \delta(\omega - \omega_0) \\ \text{If we let } \omega_0 = 1, \omega = -\Omega, \text{ we get } \int_{-\infty}^{\infty} \exp[j\Omega t] dt = 2\pi \delta(-\Omega) \\ = 2\pi \delta(\Omega)$$

Replacing Ω by ω and using Euler's rule gives

$$\int_{-\infty}^{\infty} [\cos \omega t + j \sin \omega t] dt = 2\pi \delta(\omega)$$

You can think of the integral above as the sum of an infinite number of sines and cosines of various frequencies. At any t , except $t=0$, as many of these terms will be positive as will be negative and hence their contributions will add up to zero. At $t=0$ however, all the cosine terms are $+1$, and their contributions add up to produce a spike.

$$4.8 \quad x(t) = \text{rect}\left((t + \frac{a}{2})/a\right) - \text{rect}\left((t - \frac{a}{2})/a\right) \\ X(\omega) = a \text{sinc} \frac{\omega a}{2\pi} \exp\left[j \frac{\omega a}{2}\right] - a \text{sinc} \frac{\omega a}{2\pi} \exp\left[-j \frac{\omega a}{2}\right] \\ = a \text{sinc} \frac{\omega a}{2\pi} \left[2j \sin \frac{\omega a}{2}\right]$$

Since $y(t) = \int_{-\infty}^t x(\tau) d\tau$, and $x(0) = 0$,

$$Y(\omega) = \frac{X(\omega)}{j\omega} = \frac{2a}{\omega} \operatorname{sinc}\left(\frac{\omega a}{2\pi}\right) \sin\left(\frac{\omega a}{2}\right)$$

4.9 (a) $X(\omega) = \frac{1}{2+j\omega}$; $|X(\omega)|^2 = \frac{1}{4+\omega^2}$

$$E = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1}{4+\omega^2} d\omega = \frac{1}{2\pi} \tan^{-1} \frac{\omega}{2} \Big|_0^{\infty} = \frac{1}{4}$$

$$\begin{aligned} \text{(b)} \quad X(\omega) &= \int_0^5 e^{-j\omega t} dt = \frac{1 - e^{-j\omega 5}}{j\omega} \\ &= e^{-j\frac{5}{2}\omega} \cdot \frac{2 \sin\left(\frac{5\omega}{2}\right)}{j\omega} \end{aligned}$$

$$\therefore |X(\omega)|^2 = \frac{4 \sin^2\left(\frac{5\omega}{2}\right)}{\omega^2} = \frac{2(1 - \cos 5\omega)}{\omega^2}$$

$$\begin{aligned} \therefore E &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{2(1 - \cos 5\omega)}{\omega^2} d\omega \\ &= \frac{2}{\pi} \int_0^{\infty} \frac{(1 - \cos 5\omega)}{\omega^2} d\omega = \frac{2}{\pi} \cdot \frac{5\pi}{2} = 5 \end{aligned}$$

$$\text{(c)} \quad x(t) = \Delta\left(\frac{t}{4}\right) = \frac{1}{4} \operatorname{rect}\left(\frac{t}{4}\right) * \operatorname{rect}\left(\frac{t}{4}\right)$$

$$\therefore X(\omega) = \frac{1}{4} \left(\frac{2 \sin \omega}{\omega} \right)^2 = \frac{\sin^2 \omega}{\omega^2} = \frac{1 - \cos 2\omega}{2\omega^2}$$

$$E = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[\frac{1 - \cos 2\omega}{2\omega^2} \right]^2 d\omega = 4$$

$$\text{(d)} \quad X(\omega) = \operatorname{rect}\left(\frac{\omega}{2\pi}\right)$$

$$\therefore E = \frac{1}{2\pi} \int_{-\pi}^{\pi} d\omega = 1$$

$$4.10 \quad X(t) = A \exp[-at^2] \quad y(t) = X^2(t)$$

$$(a) \quad X(\omega) = A \sqrt{\frac{\pi}{a}} \exp[-\omega^2/4a]$$

$$\begin{aligned} y(t) = X^2(t) &\leftrightarrow Y(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left(A \sqrt{\frac{\pi}{a}} \right) \exp\left[-\frac{\tau^2}{4a}\right] \left(A \sqrt{\frac{\pi}{a}} \right) \exp\left[-\frac{(\omega-\tau)^2}{4a}\right] d\tau \\ &= \frac{A}{2a} \exp\left[-\frac{\omega^2}{8a}\right] \int_{-\infty}^{\infty} \exp\left[-\frac{(\tau-\frac{\omega}{2})^2}{2a}\right] d\tau \end{aligned}$$

$$\text{using } \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[-\frac{x^2}{2\sigma^2}\right] dx = 1$$

$$Y(\omega) = \frac{A}{2a} \cdot \exp[-\omega^2/8a] \cdot \sqrt{2\pi a} = A \sqrt{\frac{\pi}{2a}} \exp[-\omega^2/8a]$$

$$(b) \quad z(t) = y(t) * h(t) \Rightarrow Z(\omega) = Y(\omega) H(\omega)$$

$$Z(\omega) = A \sqrt{\frac{\pi}{2a}} \exp[-\omega^2/8a] \cdot B \sqrt{\frac{\pi}{a}} \exp[-\omega^2/4b]$$

$$= \frac{AB\pi}{\sqrt{2ab}} \exp\left[-\omega^2\left(\frac{1}{8a} + \frac{1}{4b}\right)\right]$$

$$z(t) = \frac{AB\pi}{\sqrt{2ab}} \cdot \sqrt{\frac{ab}{\pi(a+b/2)}} \exp\left[-\frac{ab}{a+b/2} t^2\right]$$

$$= AB \sqrt{\frac{\pi}{2a+b}} \exp\left[-\frac{ab}{a+b/2} t^2\right]$$

$$(c) \quad X(t) \xrightarrow{\quad} \boxed{h(t)} \xrightarrow{y(t)} \boxed{(\quad)^2} \rightarrow Z(t)$$

$$Y(\omega) = H(\omega) X(\omega) = AB \sqrt{\frac{\pi^2}{ab}} \left(\exp[-\omega^2/4a - \omega^2/4b] \right)$$

$$= AB \sqrt{\frac{\pi^2}{ab}} \exp \left[-\frac{(a+b)}{4ab} \omega^2 \right]$$

$$y(t) = AB \sqrt{\frac{\pi}{a+b}} \exp \left[-\frac{ab}{a+b} t^2 \right]$$

$$Z(t) = (AB)^2 \frac{\pi}{a+b} \exp \left[-\frac{2ab}{a+b} t^2 \right]$$

4.11 (a) let $X(t) = \text{rect}(t/2) \longleftrightarrow X(\omega) = \frac{2 \sin \omega}{\omega}$

$$X(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{2 \sin \omega}{\omega} \exp[j\omega t] d\omega$$

$$x(0) = 1 = \frac{2}{2\pi} \int_{-\infty}^{\infty} \frac{\sin \omega}{\omega} d\omega \quad \text{or} \quad \int_{-\infty}^{\infty} \frac{\sin \omega}{\pi \omega} d\omega = 1$$

(b) $x(t) = \exp[-t^2/4\pi] \longleftrightarrow X(\omega) = 2\pi \exp[-\pi\omega^2]$

$$X(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} (2\pi \exp[-\pi\omega^2]) \exp[j\omega t] d\omega$$

$$x(0) = 1 = \int_{-\infty}^{\infty} \exp[-\pi\omega^2] d\omega$$

(c) $x(t) = \exp[-a|t|] \longleftrightarrow X(\omega) = \frac{2a}{a^2 + \omega^2}$

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left(\frac{2a}{a^2 + \omega^2} \right) \exp[j\omega t] d\omega$$

$$x(0) = 1 = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{2a}{a^2 + \omega^2} d\omega$$

(d) $x(t) = \Delta(t/T) \longleftrightarrow X(\omega) = T \left(\frac{\sin \omega T/2}{\omega T/2} \right)^2$

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} T \left(\frac{\sin \frac{\omega T}{2}}{\frac{\omega T}{2}} \right)^2 \exp[j\omega t] d\omega$$

$$x(0) = 1 = \frac{1}{2\pi} \int_{-\infty}^{\infty} T \left(\frac{\sin \frac{\omega T}{2}}{\frac{\omega T}{2}} \right)^2 d\omega$$

$$4.12 (a) \quad x(t) = y(t) = \frac{1}{a^2 + t^2}, \quad X(\omega) = Y(\omega) = \frac{\pi}{a} \exp[-a|\omega|]$$

$$\begin{aligned} \int_0^{\infty} \left(\frac{1}{a^2 + t^2} \right)^2 dt &= \frac{1}{2} \int_{-\infty}^{\infty} \left(\frac{1}{a^2 + t^2} \right)^2 dt \\ &= \frac{1}{4\pi} \int_{-\infty}^{\infty} \left(\frac{\pi}{a} \right)^2 \exp[-2a|\omega|] d\omega \\ &= \frac{\pi}{4a^2} \cdot \frac{1}{a} = \frac{\pi}{4a^3} \end{aligned}$$

$$(b) \quad x(t) = \frac{1}{a^2 + t^2}, \quad X(\omega) = \frac{\pi}{a} \exp[-a|\omega|]$$

$$y(t) = \sin ct, \quad Y(\omega) = \text{rect}(\omega/2\pi)$$

$$\begin{aligned} \int_0^{\infty} \frac{\sin ct}{a^2 + t^2} dt &= \frac{1}{2} \int_{-\infty}^{\infty} \frac{\sin ct}{a^2 + t^2} dt \\ &= \frac{1}{4\pi} \int_{-\infty}^{\infty} \frac{\pi}{a} \exp[-a|\omega|] \text{rect}(\omega/2\pi) d\omega \\ &= \frac{1}{4\pi} \int_{-\pi}^{\pi} \frac{\pi}{a} \exp[-a|\omega|] d\omega \\ &= \frac{1 - \exp[-\pi a]}{2a^2} \end{aligned}$$

$$(c) \quad x(t) = \frac{\sin^2 t}{t^2}, \quad X(\omega) = \pi \Delta(\omega/2)$$

$$Y(\omega) = \frac{\sin t}{t}, \quad Y(\omega) = \pi \text{rect}(\omega/2)$$

$$\begin{aligned} \int_{-\infty}^{\infty} \frac{\sin^3 t}{t^3} dt &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \pi^2 \Delta(\omega/2) \text{rect}(\omega/2) d\omega \\ &= \frac{\pi}{2} \int_{-1}^1 \Delta(\omega/2) d\omega \\ &= \frac{3\pi}{4} \end{aligned}$$

$$(d) \quad x(t) = \frac{\sin^2 t}{t^2}, \quad X(\omega) = \pi \Delta(\omega/2)$$

$$y(t) = \frac{\sin^2 t}{t^2}, \quad Y(\omega) = \pi \Delta(\omega/2)$$

$$\begin{aligned} \int_{-\infty}^{\infty} \frac{\sin^4 t}{t^2} dt &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \pi^2 [\Delta(\omega/2)]^2 d\omega \\ &= \frac{1}{\pi} \int_0^2 (1 - \frac{\omega}{2})^2 d\omega \\ &= \frac{2\pi}{3} \end{aligned}$$

$$4.13 \quad x(t) = \exp[-\epsilon t] u(t)$$

$$(a) \quad X(\omega) = \int_0^{\infty} \exp[-\epsilon t] \exp[-j\omega t] dt = \frac{1}{\epsilon + j\omega}$$

$$(b) \quad X(\omega) = \frac{\epsilon - j\omega}{\epsilon^2 + \omega^2} = \frac{\epsilon}{\epsilon^2 + \omega^2} - j \frac{\omega}{\epsilon^2 + \omega^2}$$

$$(c) \quad \lim_{\epsilon \rightarrow 0} \frac{\epsilon}{\epsilon^2 + \omega^2} = \begin{cases} 0 & \omega \neq 0 \\ \infty & \omega = 0 \end{cases} \quad \text{and} \quad \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\epsilon}{\epsilon^2 + \omega^2} d\omega = 1$$

so $\lim_{\epsilon \rightarrow 0} \frac{\epsilon}{\epsilon^2 + \omega^2}$ is an impulse in ω

$$\begin{aligned} F \left\{ \lim_{\epsilon \rightarrow 0} \exp[-\epsilon t] u(t) \right\} &= \lim_{\epsilon \rightarrow 0} \left(\frac{\epsilon}{\epsilon^2 + \omega^2} \right) - \lim_{\epsilon \rightarrow 0} j \frac{\omega}{\omega^2 + \epsilon^2} \\ &= \pi \delta(\omega) + \frac{1}{j\omega} \end{aligned}$$

$$4.14 \quad H(\omega) = \text{rect} \left(\frac{\omega}{4} \right); \quad X(\omega) = \frac{1}{\alpha + j\omega}$$

$$(a) \quad Y(\omega) = H(\omega)X(\omega) = \begin{cases} \frac{1}{\alpha + j\omega} & -2 \leq \omega \leq 2 \\ 0 & \text{elsewhere} \end{cases}$$

$$(b) \quad \text{Input signal energy} = \frac{1}{2\alpha}$$

$$\text{Output signal energy} = \frac{1}{2\pi} \int_{-2}^2 \frac{1}{\alpha^2 + \omega^2} d\omega$$

$$= \frac{1}{\pi} \int_0^2 \frac{1}{\alpha^2 + \omega^2} d\omega = \frac{1}{\pi\alpha} \tan^{-1} \frac{2}{\alpha}$$

$$\therefore \frac{1}{\pi \alpha} \tan^{-1} \frac{2}{\alpha} = \frac{1}{4 \alpha} \quad \text{or } \alpha = 2.$$

$$\begin{aligned} 4.15 \quad (a) \quad \mathcal{F}\{x(-2t+1)\} &= \frac{1}{2} e^{-j\frac{\omega}{2}} \times \left(-\frac{\omega}{2}\right) \\ &= \frac{1}{2} e^{-j\frac{\omega}{2}} \cdot \frac{\frac{\omega^2}{4} - j2\omega + 2}{-\frac{\omega^2}{4} - j2\omega + 3} \end{aligned}$$

$$(b) \quad \mathcal{F}\{x(t)e^{-jt}\} = x(\omega+1) = \frac{(\omega+1)^2 + j4(\omega+1) + 2}{-(\omega+1)^2 + j4(\omega+1) + 3}$$

$$(c) \quad \mathcal{F}\left\{\frac{dx(t)}{dt}\right\} = j\omega x(\omega) = j\omega \frac{\omega^2 + j4\omega + 2}{-\omega^2 + j4\omega + 3}$$

$$\begin{aligned} (d) \quad \mathcal{F}\{x(t)\sin\pi t\} &= \frac{\pi}{j} [x(\omega-\pi) - x(\omega+\pi)] \\ &= \frac{\pi}{j} \left\{ \frac{(\omega-\pi)^2 + j4(\omega-\pi) + 2}{-(\omega-\pi)^2 + j4(\omega-\pi) + 3} - \frac{(\omega+\pi)^2 + j4(\omega+\pi) + 2}{-(\omega+\pi)^2 + j4(\omega+\pi) + 3} \right\} \end{aligned}$$

$$\begin{aligned} (e) \quad \mathcal{F}\{x(t) * \delta(t-1)\} &= e^{-j\omega} x(\omega) \\ &= e^{-j\omega} \frac{\omega^2 + j4\omega + 2}{-\omega^2 + j4\omega + 3} \end{aligned}$$

$$\begin{aligned} (f) \quad \mathcal{F}\{x(t) * x(t-1)\} &= e^{-j\omega} [x(\omega)]^2 \\ &= e^{-j\omega} \left[\frac{\omega^2 + j4\omega + 2}{-\omega^2 + j4\omega + 3} \right]^2 \end{aligned}$$

$$4.16 \quad (a) \quad \mathcal{F}\left\{\frac{\sin \alpha t}{\pi t}\right\} = \text{rect}(\omega/2\alpha)$$

$$\begin{aligned} \mathcal{F}\left\{x(t) * \frac{\sin \alpha t}{\pi t}\right\} &= X(\omega) \cdot \text{rect}(\omega/2\alpha) \\ &= X(\omega), \quad \alpha > \omega_c \end{aligned}$$

$$\text{hence } x(t) * \frac{\sin \alpha t}{\pi t} = x(t), \quad \alpha > \omega_c$$

$$(b) \quad x(t) = \frac{\sin t}{t} \Rightarrow X(\omega) = \text{rect}(\omega/2), \quad \omega_c = 1$$

$$\int_{-\infty}^{\infty} \frac{\sin \alpha \tau}{\tau} \frac{\sin(t-\tau)}{\pi(t-\tau)} d\tau = \begin{cases} x(t) & , \alpha \geq 1 \\ \mathcal{F}^{-1}\{\text{rect}(\omega/2)\text{rect}(\omega/2\alpha)\} & \alpha \leq 1 \end{cases}$$

$$= \begin{cases} \frac{\sin t}{t} & , \alpha \geq 1 \\ \frac{\sin \alpha t}{t} & , |\alpha| \leq 1 \end{cases}$$

4.17 For an inductor, $V_L(t) = L \frac{di_L(t)}{dt}$

$$\Rightarrow V_L(\omega) = L(j\omega) I_L(\omega)$$

$$\text{the inductance} = \frac{V_L(\omega)}{I_L(\omega)} = j\omega L$$

For a capacitor $i_C(t) = C \frac{dv_C(t)}{dt}$

$$I_C(\omega) = j\omega C V_C(\omega)$$

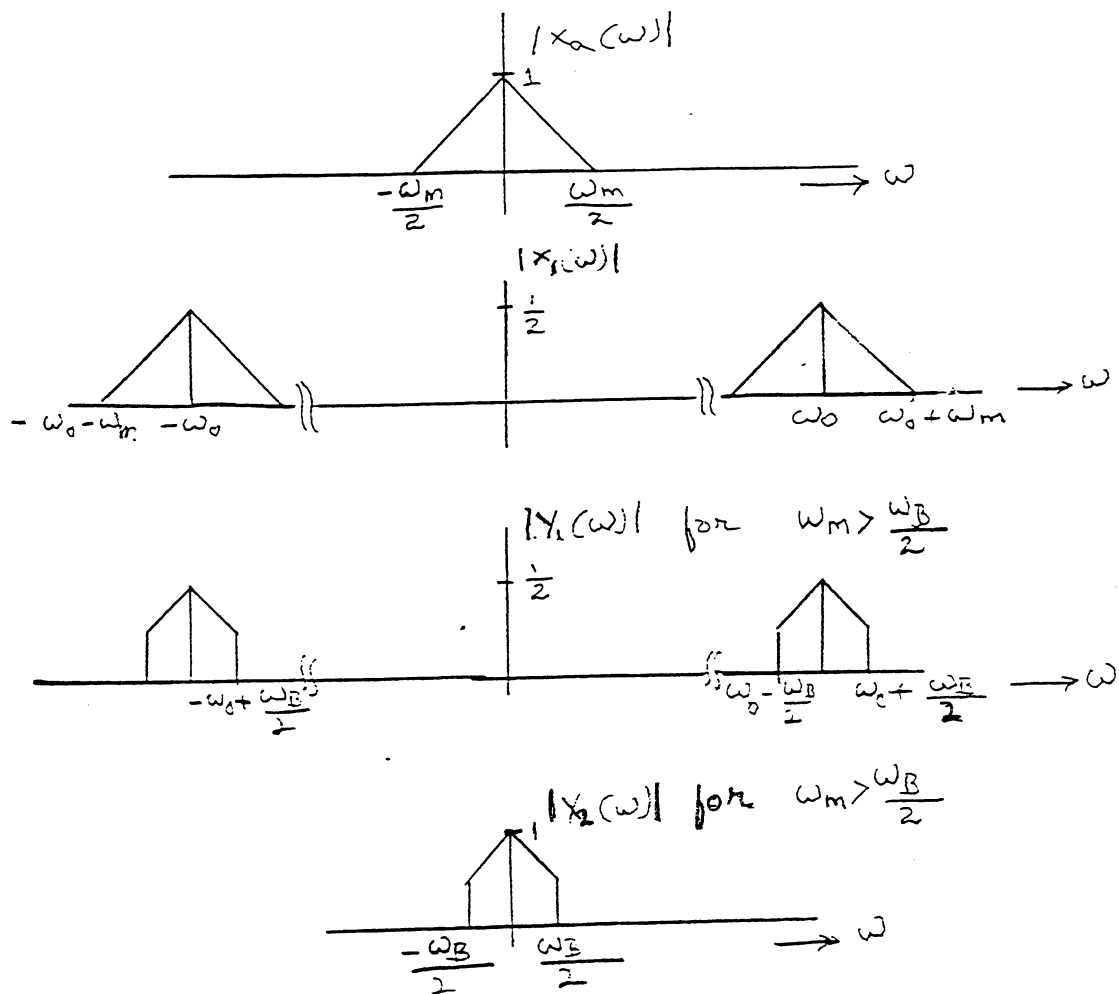
$$\text{the capacitance} = \frac{V_C(\omega)}{I_C(\omega)} = \frac{1}{j\omega C}$$

4.18 (a) $H(\omega) = \frac{1/j\omega C}{R + 1/j\omega C} = \frac{1}{jRC\omega + 1} = \frac{1}{1 + j10\omega}$

$H(\omega)$ is a low-pass filter.

(b) $H(\omega) = \frac{R}{R + 1/j\omega C} = \frac{j10\omega}{1 + j10\omega}$

4.19 Let $y_1(t)$ and $y_2(t)$ be the inputs to the systems with impulse responses $h_1(t)$ and $h_2(t)$. The figures below show the spectra of the various signals in the system, where $y(t)$ is the output of the



Since $H_2(\omega) = \text{rect}\left(\frac{\omega}{\omega_B}\right)$, $Y(\omega) = X_2(\omega)$.

When $\omega_m < \frac{\omega_B}{2}$, $Y_1(\omega) = X_1(\omega)$ and $Y_2(\omega) = X_a(\omega)$

$$A.20 \quad x(t) = \text{Sa}\left(\frac{3\omega_B t}{2}\right) \longleftrightarrow X(\omega) = \frac{2\pi}{3\omega_B} \text{rect}\left(\frac{\omega}{3\omega_B}\right)$$

$$y(t) = \frac{\pi}{3\omega_B} \text{Sa}\left(\frac{\omega_B t}{2}\right)$$

$$\begin{aligned}
 4.21 \quad (a) \quad H(\omega) &= \int_{-\infty}^{\infty} \frac{e^{-j\omega t}}{\pi t} dt \\
 &= \int_{-\infty}^0 \frac{e^{-j\omega t}}{\pi t} dt + \int_0^{\infty} \frac{e^{-j\omega t}}{\pi t} dt \\
 &= \int_0^{\infty} \frac{e^{-j\omega t} - e^{j\omega t}}{\pi t} dt = \int_0^{\infty} -2j \frac{\sin \omega t}{\pi t} dt \\
 &= -\frac{2j}{\pi} \times \frac{\pi}{2} \operatorname{sgn} \omega = -j \operatorname{sgn} \omega
 \end{aligned}$$

$$\begin{aligned}
 (b) \quad X(\omega) &= \pi [\delta(\omega - \omega_0) + \delta(\omega + \omega_0)] \\
 Y(\omega) &= \mathcal{F}\{\text{Hilbert transform of } x(t)\} \\
 &= (-j \operatorname{sgn} \omega) \pi [\delta(\omega - \omega_0) + \delta(\omega + \omega_0)] \\
 &= \frac{\pi}{j} [\delta(\omega + \omega_0) - \delta(\omega - \omega_0)] \\
 \therefore y(t) &= \sin \omega_0 t
 \end{aligned}$$

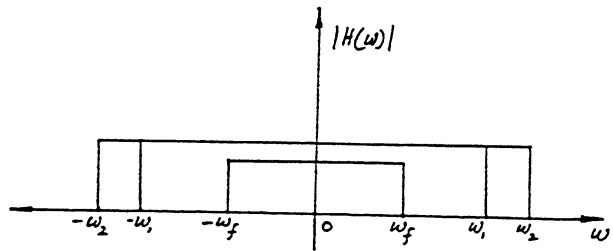
$$\begin{aligned}
 4.22 \quad \mathcal{F}\{R_x(t)\} &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x(\tau) x(\tau+t) \exp[-j\omega t] dt d\tau \\
 &= \int_{-\infty}^{\infty} x(\tau) \int_{-\infty}^{\infty} x(\tau+t) \exp[-j\omega t] dt d\tau \\
 &= X(\omega) \int_{-\infty}^{\infty} x(\tau) \exp[j\omega \tau] d\tau \\
 &= |X(\omega)|^2, \quad \text{for real-valued } x(t)
 \end{aligned}$$

$$\begin{aligned}
 4.23 \quad R_y(t) &= \int_{-\infty}^{\infty} y(\tau) y(\tau+t) d\tau \\
 &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(u) x(\tau-u) h(v) x(\tau+t-v) du dv d\tau \\
 &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(u) h(v) R_x(t-v+u) du dv \\
 &= R_x(t) * h(t) * h(-t)
 \end{aligned}$$

$$4.24 \quad x(t) = w_1 S_a(w_1 t) + w_2 S_a(w_2 t) \quad w_1 < w_2$$

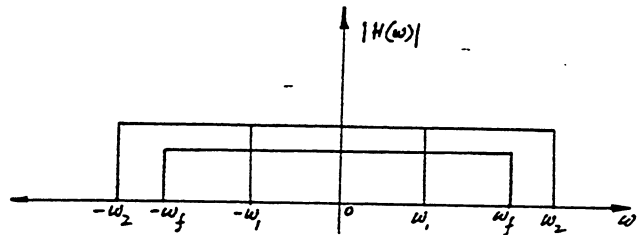
$$\longleftrightarrow X(\omega) = \pi \text{rect}(\omega/2w_1) + \pi \text{rect}(\omega/2w_2)$$

$$(a) \quad 0 < w_f < w_1 < w_2$$



$$\begin{aligned}
 y(t) &= F^{-1}[H(\omega) X(\omega)] \\
 &= F^{-1}[\pi \text{rect}(\omega/2w_1) + \pi \text{rect}(\omega/2w_2)] \quad -w_f \leq \omega \leq w_f \\
 &= F^{-1}[2\pi \text{rect}(\omega/2w_f)] \\
 &= 2w_f S_a(w_f t) \\
 &= 2 \frac{\sin w_f t}{t}
 \end{aligned}$$

$$(b) \quad w_1 < w_f < w_2$$



$$\begin{aligned}
 y(t) &= F^{-1}[H(\omega) X(\omega)] \\
 &= F^{-1}[\text{rect}(\omega/2w_f) \cdot \pi \text{rect}(\omega/2w_1) + \text{rect}(\omega/2w_f) \cdot \pi \text{rect}(\omega/2w_2)] \\
 &= F^{-1}[\pi \text{rect}(\omega/2w_1) + \pi \text{rect}(\omega/2w_f)]
 \end{aligned}$$

$$= W_1 S_a(W_1 t) + W_f S_a(W_f t)$$

$$= \frac{\sin W_1 t}{t} + \frac{\sin W_f t}{t}$$

$$(c) \quad W_1 < W_2 < W_f$$

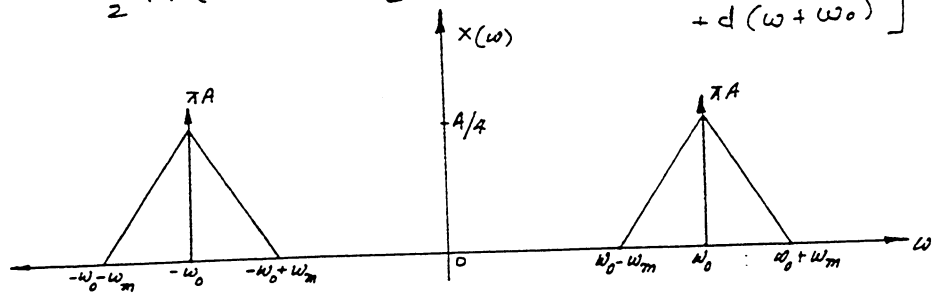
from (a), (b) we conclude that

$$y(t) = \frac{\sin W_1 t}{t} + \frac{\sin W_2 t}{t}$$

$$4.25 \quad x(t) = (m(t) + A) \cos \omega_0 t$$

$$(a) \quad X(\omega) = F[m(t) \cos \omega_0 t + A \cos \omega_0 t]$$

$$= \frac{1}{2} M(\omega + \omega_0) + \frac{1}{2} M(\omega - \omega_0) + A \pi [\delta(\omega - \omega_0) + \delta(\omega + \omega_0)]$$

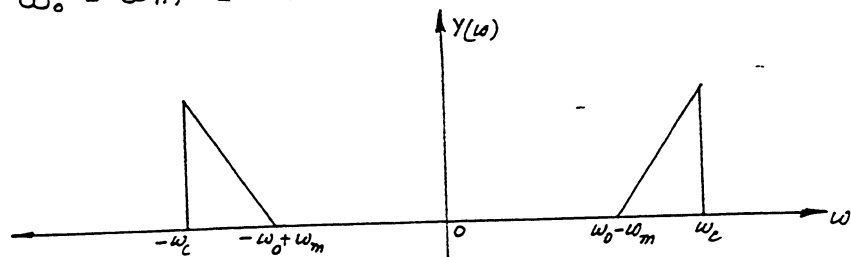


$$(b) \quad y(t) = h_c(t) * x(t) \Rightarrow Y(\omega) = H_c(\omega) X(\omega)$$

$$(i) \quad 0 \leq \omega_c \leq \omega_0 - \omega_m$$

$$y(t) = 0$$

$$(ii) \quad \omega_0 - \omega_m \leq \omega_c < \omega_0$$



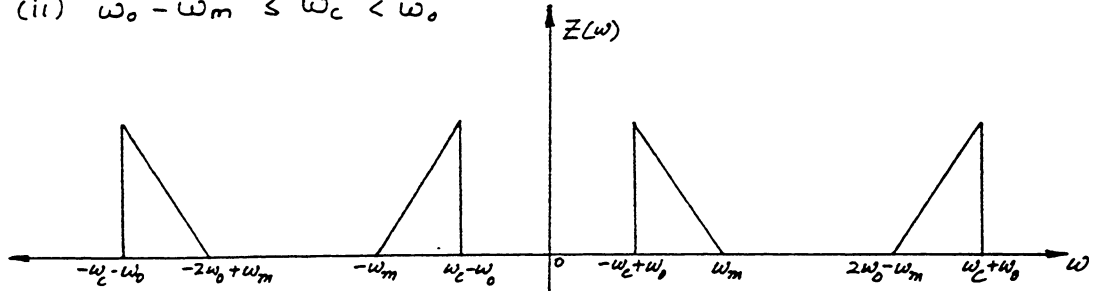
$$(iii) \quad \omega_c > \omega_0 + \omega_m$$

$$Y(\omega) = X(\omega), \quad \text{see part (a)}$$

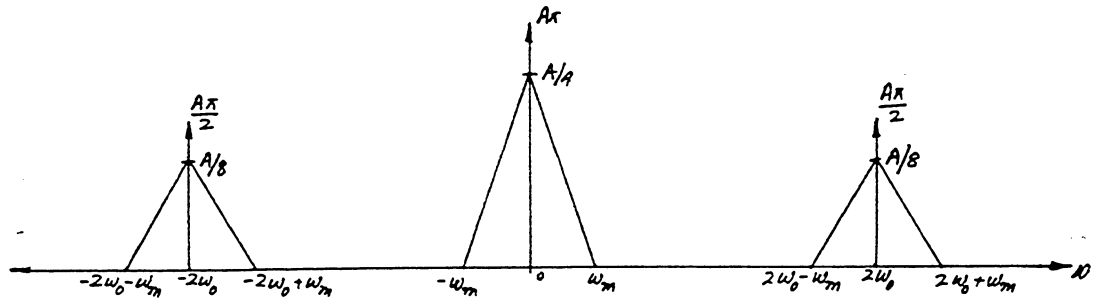
$$(c) \quad Z(t) = y(t) \cos \omega_0 t \Rightarrow Z(\omega) = \frac{1}{2} Y(\omega - \omega_0) + \frac{1}{2} Y(\omega + \omega_0)$$

$$(i) \quad 0 \leq \omega_c < \omega_0 - \omega_m, \quad Y(\omega) = 0 \Rightarrow Z(\omega) = 0$$

$$(ii) \quad \omega_0 - \omega_m \leq \omega_c < \omega_0$$

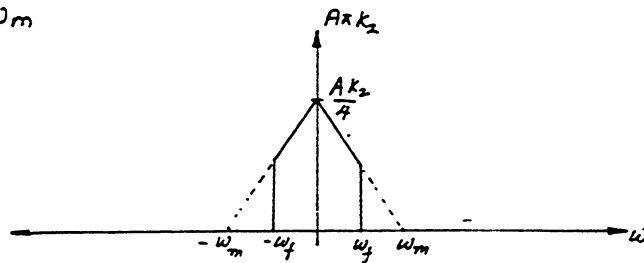


$$(iii) \quad \omega_c > \omega_0 + \omega_m$$

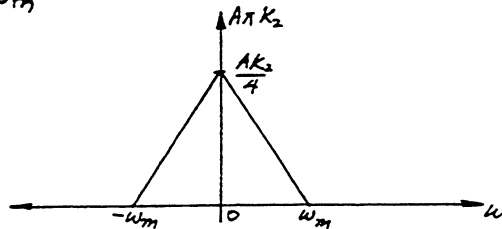


$$(d) \quad V(t) = h_f(t) * Z(t) \Rightarrow V(\omega) = H_f(\omega) Z(\omega), \quad \omega_c > \omega_0 + \omega_m$$

$$(i) \quad \omega_f < \omega_m$$



$$(ii) \quad \omega_f < 2\omega_0 - \omega_m$$



(iii) For $\omega_f > 2\omega_0 + \omega_m$, $V(\omega) = Z(\omega)$ in (c)-(ii) and the spectrum is the same as in (c)-(ii)

$$4.26 \quad z(t) = [x(t) A \cos \omega_0 t] [A \cos(\omega_0 t + \theta)] \\ = \frac{A^2}{2} x(t) [\cos \theta + \cos(2\omega_0 t + \theta)]$$

If we pass $z(t)$ through a low-pass filter with cutoff ω_m

$$\hat{x}(t) = \frac{A^2}{2} \cos \theta x(t)$$

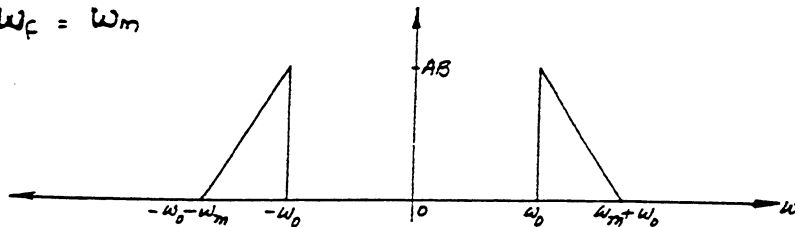
Desired output is $x_d(t) = x(t)$

$$\therefore \hat{x}(t) = \frac{A^2}{2} \cos \theta x_d(t)$$

$$4.27 (a) \quad x(t) = m(t) \cos \omega_0 t \Rightarrow X(\omega) = \frac{1}{2} M(\omega + \omega_0) + \frac{1}{2} M(\omega - \omega_0)$$

$$y(t) = h_f(t) * x(t) \Rightarrow Y(\omega) = H_f(\omega) X(\omega)$$

for $\omega_f = \omega_m$

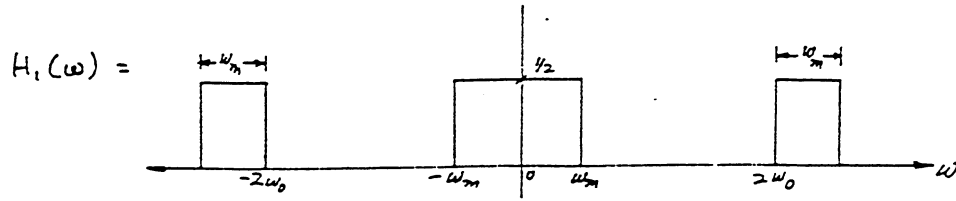


$$(b) \quad h_f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} H_f(\omega) \cdot \exp[j\omega t] d\omega \\ = \frac{1}{2\pi} \int_{-\omega_0 + \omega_m}^{\omega_0} B \exp[j\omega t] d\omega + \frac{1}{2\pi} \int_{\omega_0}^{\omega_m + \omega_0} B \exp[j\omega t] d\omega \\ = \frac{B}{2\pi} \int_{\omega_0}^{\omega_0 + \omega_m} \exp[-j\omega t] d\omega + \frac{B}{2\pi} \int_{\omega_0}^{\omega_0 + \omega_m} \exp[j\omega t] d\omega \\ = \frac{B}{\pi t} [\sin(\omega_0 + \omega_m)t - \sin \omega_0 t]$$

$$h_f(t) \neq 0 \quad \text{for } t < 0$$

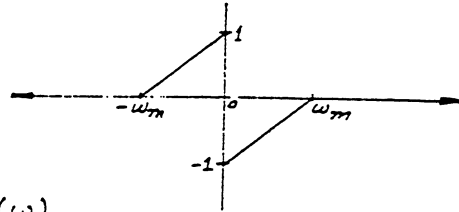
implies the filter is an unrealizable

$$4.28 \quad y(t) = [h_1(t) * m(t)] \cos \omega_0 t + [h_2(t) * m(t)] \sin \omega_0 t$$



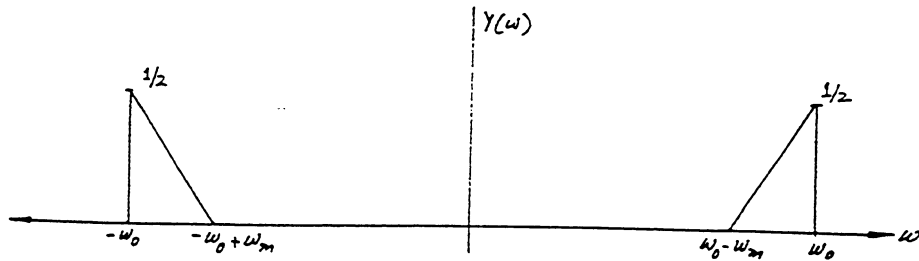
$$\begin{aligned} H_2(\omega) M(\omega) &= \frac{-1}{2j} [H_0(\omega - \omega_0) - H_0(\omega + \omega_0)] M(\omega) \\ &= \frac{-1}{2j} [H_0(\omega - \omega_0) M(\omega) - H_0(\omega + \omega_0) M(\omega)] \end{aligned}$$

where $H_0(\omega - \omega_0) M(\omega) - H_0(\omega + \omega_0) M(\omega) = F(\omega)$



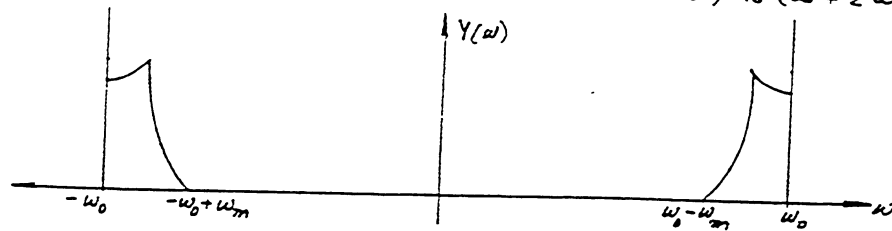
$$H_1(\omega) M(\omega) = \frac{1}{2} M(\omega)$$

$$Y(\omega) = \frac{1}{4} [M(\omega - \omega_0) + M(\omega + \omega_0)] + \frac{1}{4} [F(\omega - \omega_0) - F(\omega + \omega_0)]$$



$$\begin{aligned} (b) \quad Y(\omega) &= H_1(\omega) M(\omega) * \pi [\delta(\omega - \omega_0) + \delta(\omega + \omega_0)] + [H_2(\omega) M(\omega)] \\ &\quad * -j\pi [\delta(\omega - \omega_0) - \delta(\omega + \omega_0)] \\ &= \frac{\pi}{2\pi} \left\{ H_1(\omega - \omega_0) M(\omega - \omega_0) + H_1(\omega + \omega_0) M(\omega + \omega_0) \right\} \\ &\quad - j\frac{\pi}{2\pi} [H_2(\omega - \omega_0) M(\omega - \omega_0) - H_2(\omega + \omega_0) M(\omega + \omega_0)] \\ &= \frac{1}{2} \left\{ \frac{1}{2} [H_0(\omega - 2\omega_0) + H_0(\omega)] M(\omega - \omega_0) \right\} + \frac{1}{2} [H_0(\omega) + H_0(\omega + 2\omega_0)] \\ &\quad M(\omega + \omega_0) - \frac{j}{2} \left\{ -\frac{1}{2j} [(H_0(\omega - 2\omega_0) - H_0(\omega)) M(\omega - \omega_0) \right. \\ &\quad \left. - \frac{1}{2j} [H_0(\omega) - H_0(\omega + 2\omega_0)] M(\omega + \omega_0) \right\} \end{aligned}$$

$$Y(\omega) = \frac{1}{2} [M(\omega - \omega_0) H_0(\omega - 2\omega_0) + M(\omega + \omega_0) H_0(\omega + 2\omega_0)]$$



We obtain a distorted LSB signal. The conclusion is: as long as $H_0(\omega - \omega_0) + H_0(\omega + \omega_0) = \text{constant}$ in the frequency band of the signal, we can obtain undistorted LSB.

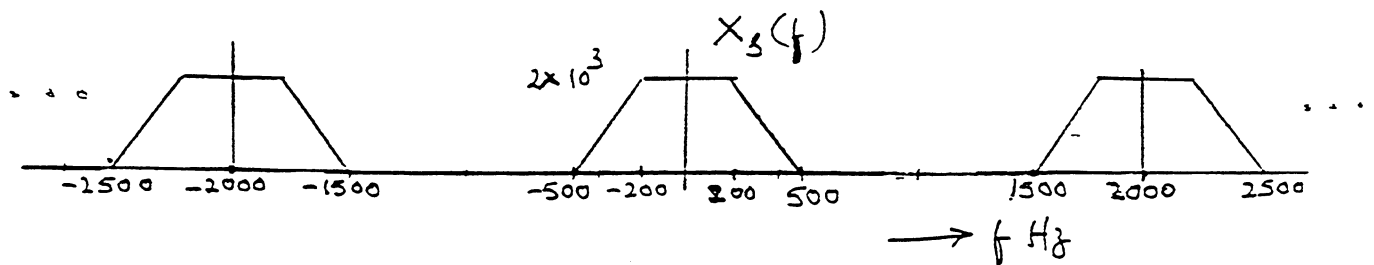
1.29 Bandwidth of $z(t)$ is 500 Hz

(a) $T_{\max} = \frac{1}{2 \times 500} = 1 \text{ ms}$

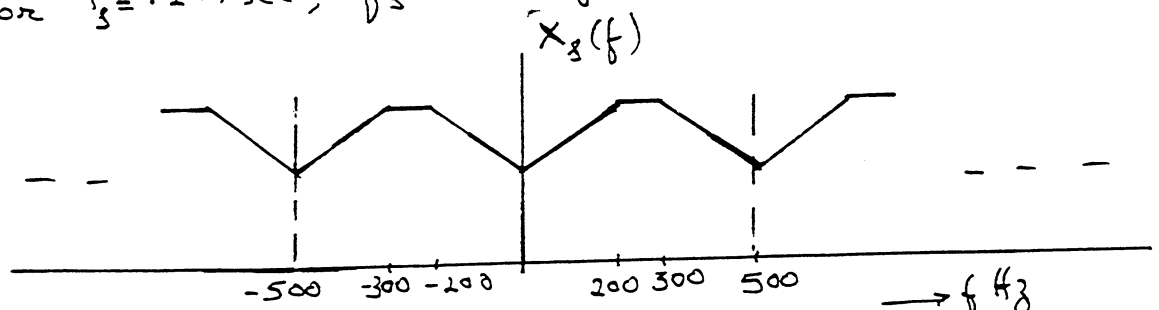
(b) $x(\omega) = \text{rect}\left(\frac{\omega}{600\pi}\right)$, $y(\omega) = \text{rect}\left(\frac{\omega}{1400\pi}\right)$

$$\therefore Z(\omega) = x(\omega) * y(\omega) = \begin{cases} 1000\pi + \omega & -1000\pi < \omega < -400\pi \\ 600\pi & -400\pi < \omega < 400\pi \\ 1000\pi - \omega & 400\pi < \omega < 1000\pi \\ 0 & \text{otherwise} \end{cases}$$

For $T_s = 0.5 \text{ msec}$, $f_s = \frac{1}{T_s} = 2000 \text{ Hz}$



For $T_s = 0.2 \text{ msec}$, $f_s = 500 \text{ Hz}$



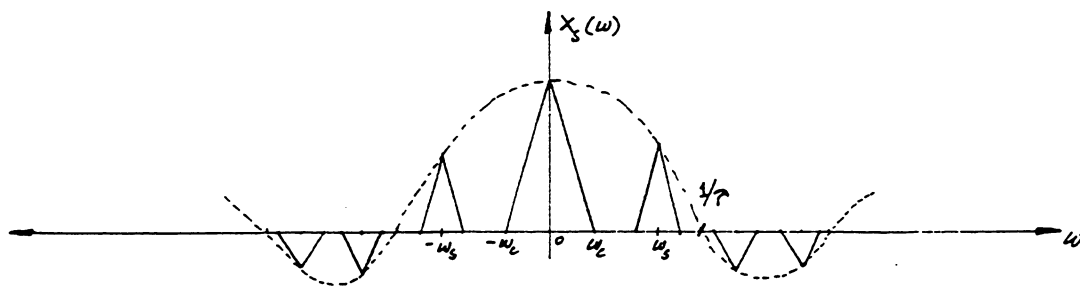
4.30 (a) The pulse train $p(t)$ has the Fourier series expansion

$$p(t) = \frac{\tau}{T} \sum_{k=-\infty}^{\infty} \frac{\sin(k\omega_s \tau/2)}{k\omega_s \tau/2} \exp[jk\omega_s t], \quad \omega_s = \frac{2\pi}{T}$$

Its spectrum

$$P(\omega) = \frac{2\pi\tau}{T} \sum_{k=-\infty}^{\infty} \frac{\sin(k\omega_s \tau/2)}{k\omega_s \tau/2} \delta(\omega - k\omega_s)$$

$$\begin{aligned} X_s(\omega) &= \frac{1}{2\pi} X(\omega) * P(\omega) \\ &= \frac{\tau}{T} \sum_{k=-\infty}^{\infty} \frac{\sin(k\omega_s \tau/2)}{k\omega_s \tau/2} X(\omega - k\omega_s) \end{aligned}$$



(b) Yes, so long as $\omega_s \geq 2\omega_c$. By applying $X_s(t)$ to a low-pass filter of bandwidth ω_c , $x(t)$ is easily recovered without distortion.

4.31 (a) Assuming ideal rectangular pulses, the flat-top sampled signal is

$$x_s(t) = \sum_{k=-\infty}^{\infty} x(kT) \text{rect}((t - kT)/\tau)$$

$$(b) X_s(\omega) = \frac{\tau}{T} \sum_{k=-\infty}^{\infty} X(\omega - k\omega_s) \frac{\sin \omega \tau/2}{\omega \tau/2}, \quad \omega_s = \frac{2\pi}{T}$$

(c) $X_s(\omega)$ appears on the surface to be similar to part (a) of problem 4.30. There is an important difference, however. A low-pass filter operating on $X_s(\omega)$ in the flat-top sampled signal will not give distortion-free output proportional to $x(t)$.

(d) Using a low-pass filter alone, the output spectrum would be $\frac{\tau}{T_s} \frac{X(\omega) \sin(\omega\tau/2)}{\omega\tau/2}$,

which is clearly not proportional to $X(\omega)$ as needed. Therefore, $x(t)$ cannot be recovered distortion-free.

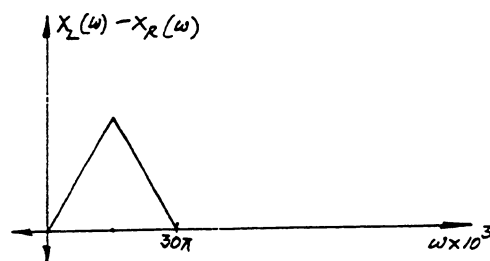
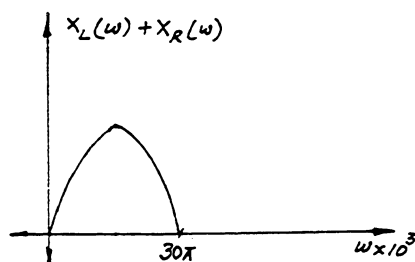
(e) The factor $Q(\omega) = \sin(\omega\tau/2)/(\omega\tau/2)$ represents distortion which may be corrected by adding a second filter, called an equalizing filter.

It must have a transfer function.

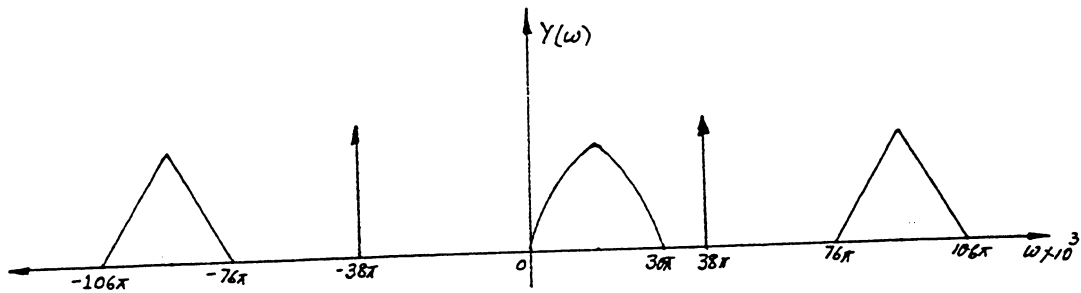
$$\begin{aligned} H_{eq}(\omega) &= 1/Q(\omega) & |\omega| < \omega_c \\ &= \frac{\omega\tau/2}{\sin(\omega\tau/2)} & |\omega| < \omega_c \\ &= \text{arbitrary} & \text{elsewhere} \end{aligned}$$

The equalized output is $X_o(\omega) = \frac{\tau}{T} X(\omega)$
and $X_o(t) = \frac{\tau}{T} x(t)$

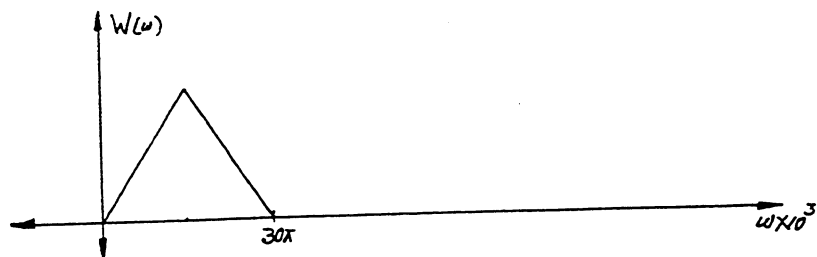
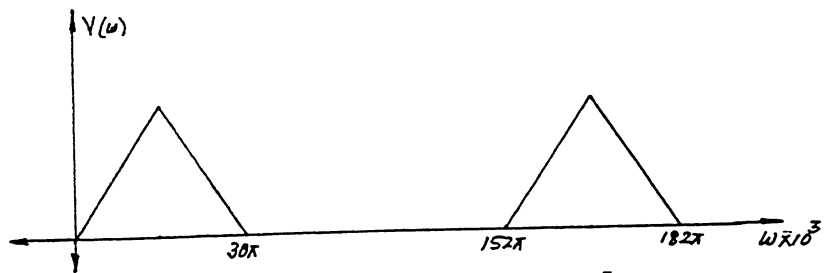
4.32



$$\begin{aligned}
 (a) \quad y(t) &= (x_L(t) - x_R(t) \cos 2\omega_1 t + x_L(t) + x_R(t) + \cos \omega_1 t) \\
 Y(\omega) &= \frac{1}{2} [X_L(\omega - 2\omega_1) - X_R(\omega - 2\omega_1) + X_L(\omega + 2\omega_1) - X_R(\omega + 2\omega_1)] \\
 &\quad + X_L(\omega) + X_R(\omega) + \frac{1}{2} \delta(\omega - \omega_1) + \frac{1}{2} \delta(\omega + \omega_1)
 \end{aligned}$$



(b)



$$(c) \quad W(\omega) = X_L(\omega) - X_R(\omega)$$

$$Z(\omega) = X_L(\omega) + X_R(\omega)$$

Then

$$X_L(\omega) = (W(\omega) + Z(\omega))/2$$

and

$$X_R(\omega) = (Z(\omega) - W(\omega))/2$$

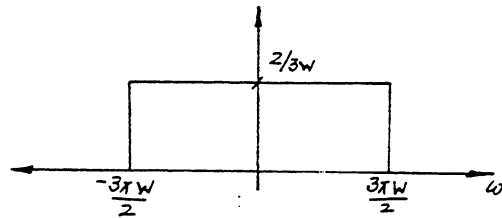
$$4.33 \quad \sum_{n=-\infty}^{\infty} \delta'(t - nT) = \sum_{n=-\infty}^{\infty} \frac{d}{dt} \delta(t - nT)$$

$$\text{but } F \left\{ \sum_{n=-\infty}^{\infty} \delta(t - nT) \right\} = \frac{2\pi}{T} \sum_{n=-\infty}^{\infty} \delta(\omega - n\frac{2\pi}{T}) = \omega_0 \sum_{n=-\infty}^{\infty} \delta(\omega - n\omega_0)$$

$$\Rightarrow \sum_{n=-\infty}^{\infty} \delta'(t - nT) \longleftrightarrow \sum_{n=-\infty}^{\infty} j n \omega_0^2 \delta(\omega - n\omega_0) = j \omega_0^2 \sum_{n=-\infty}^{\infty} n \delta(\omega - n\omega_0)$$

$$4.34 (a) \quad \text{Sinc} \left(3Wt/2 \right)$$

$$\text{Bandwidth} = \frac{3\pi W}{2}$$



$$(b) \quad \exp(-3t)u(t) = x(t)$$

$$|X(\omega)| = \frac{1}{\sqrt{9 + \omega^2}}$$

$$\text{let } \frac{|X(\omega)|^2}{|X(0)|^2} = \frac{1}{2} \Rightarrow \omega = \pm 3$$

$$\therefore \text{Bandwidth} = 3$$

$$(c) \quad \int_{-W_B}^{W_B} |X(\omega)|^2 d\omega = \int_{-W_B}^{W_B} \frac{1}{9 + \omega^2} d\omega = 0.95 \int_{-\infty}^{\infty} \frac{1}{9 + \omega^2} d\omega = 0.95 \times \frac{\pi}{3}$$

$$\Rightarrow \frac{2}{3} \tan^{-1} \left(\frac{W_B}{3} \right) = 0.95 \times \frac{\pi}{3}$$

$$W_B = 3 \tan \left(\frac{0.95 \times \pi}{2} \right)$$

$$\approx 38 \text{ rad/sec}$$

$$(d) x(t) = \sqrt{\frac{\alpha}{\pi}} \exp[-\alpha t^2] \longleftrightarrow X(\omega) = \exp[-\omega^2/4\alpha]$$

$$\int_{-\infty}^{\infty} \omega^2 |X(\omega)|^2 d\omega = \int_{-\infty}^{\infty} \omega^2 \exp[-\omega^2/2\alpha] d\omega = 2 \int_0^{\infty} \omega^2 \exp[-\omega^2/2\alpha] d\omega$$

$$= \alpha \sqrt{2\alpha\pi}$$

$$\int_{-\infty}^{\infty} |X(\omega)|^2 d\omega = 2 \int_0^{\infty} \exp[-\omega^2/2\alpha] d\omega = \sqrt{2\alpha\pi}$$

$$B_{rms}^2 = \frac{\int_{-\infty}^{\infty} \omega^2 |X(\omega)|^2 d\omega}{\int_{-\infty}^{\infty} |X(\omega)|^2 d\omega} = \frac{\alpha \sqrt{2\alpha\pi}}{\sqrt{2\alpha\pi}} = \alpha$$

$$B_{rms} = \sqrt{\alpha}$$

$$4.35(a) \quad c_n = \frac{1}{W_s} \int_{-W_s/2}^{W_s/2} X_s(\omega) \exp[-jn \frac{2\pi}{W_s} \omega] d\omega, \quad W_s = \frac{2\pi}{T}$$

Since in the central period $X_s(\omega) = X(\omega)$

$$= \frac{1}{W_s} \int_{-W_B}^{W_B} X(\omega) \exp[-jn \frac{2\pi}{W_s} \omega] d\omega$$

$$= \frac{2\pi}{W_s} x\left(-\frac{n 2\pi}{W_s}\right) = \frac{2\pi}{W_s} x(-nT)$$

$$(b) \quad X_s(\omega) = \sum_{n=-\infty}^{\infty} c_n \exp[jn \frac{2\pi}{W_s} \omega]$$

$$= \sum_{n=-\infty}^{\infty} \frac{2\pi}{W_s} x(-nT) \exp[jn \frac{2\pi}{W_s} \omega]$$

$$= \sum_{k=-\infty}^{\infty} \frac{2\pi}{W_s} x(kT) \exp[-jk \frac{2\pi}{W_s} \omega]$$

$$x_s(t) = \sum_{k=-\infty}^{\infty} \frac{2\pi}{W_s} x(kT) \delta\left(t - k \frac{2\pi}{W_s}\right)$$

$$(c) \quad X(\omega) = X_s(\omega) \text{rect}(\omega/2W_B)$$

taking the inverse Fourier transform yields

$$\begin{aligned}
 x(t) &= x_s(t) * \frac{\sin \omega_B t}{\pi t} \\
 &= \sum_{k=-\infty}^{\infty} \frac{2 \omega_B}{\omega_s} x(kT) \frac{\sin \omega_B (t-kT)}{\omega_B (t-nT)}
 \end{aligned}$$

$$4.35(a) \quad X(t) = \frac{1}{\sqrt{2\pi\Delta^2}} \exp[-t^2/2\Delta^2] \longleftrightarrow X(\omega) = \exp[-\Delta^2\omega^2/2]$$

$$T = 2 \sqrt{\frac{\Delta^2}{4\sqrt{2\pi\Delta^2}}} / \frac{1}{2\sqrt{2\pi\Delta^2}} = \sqrt{2} \Delta$$

$$X_m = 1 \quad B_{eq} = \frac{\int_0^{\infty} \exp[-\Delta^2\omega^2] d\omega}{|X(0)|^2} = \frac{\sqrt{2\pi}}{2\Delta}$$

$$\therefore B_{eq} T = \sqrt{\pi}$$

$$(b) \quad X(t) = \frac{\sin 2\pi W t}{\pi t} \longleftrightarrow X(\omega) = \text{rect}(\omega/4\pi W)$$

$$T = \frac{1}{W} \quad B = 2\pi W$$

$$\Rightarrow BT = 2\pi$$

$$(c) \quad z(t) = A e^{-\alpha t} u(t) \longleftrightarrow X(\omega) = \frac{A}{\alpha + j\omega}$$

$$T = \frac{1}{\alpha} \quad B = \alpha$$

$$\therefore BT = 1$$

CHAPTER 5

5.1 (a) $x(t) = \exp[t+1]$ does not have a bilateral Laplace transform because $\int_{-\infty}^{\infty} x(t) \exp[-st] dt$ does not converge

$$(b) X_B(s) = \int_{-\infty}^{\infty} \exp[bt] u(-t) \exp[-st] dt$$

$$= \frac{1}{b-s}, \quad \operatorname{Re}\{s\} < b$$

(c) $x(t) = |t|$ does not have a bilateral Laplace transform

(d) $x(t) = (1-|t|)$ does not have a bilateral Laplace transform.

$$(e) X_B(s) = \int_{-\infty}^{\infty} \exp[-2|t|] \exp[-st] dt$$

$$= \frac{4}{4-s^2}, \quad -2 < \operatorname{Re}\{s\} < 2$$

(f) $x(t) = t^2 \exp[-t] u[-t]$ does not have a bilateral Laplace transform

$$(g) X_B(s) = \int_{-\infty}^{\infty} \cos at u(-t) \exp[-st] dt$$

$$= \frac{-s}{s^2+a^2}, \quad \operatorname{Re}\{s\} < 0$$

$$(h) X_B(s) = \int_{-\infty}^{\infty} \sinh at u(-t) \exp[-st] dt$$

$$= \frac{2a}{a^2-s^2}, \quad \operatorname{Re}\{s\} < -a$$

5.2

$$(i) X_1(s) = \int_0^2 t e^{-st} dt = \frac{1-e^{-2s}}{s^2} - \frac{e^{-2s}}{s}$$

$$(i) X_+(s) = X_1(s) + \frac{1}{2} = \frac{1 - e^{-2s}}{s^2} - \frac{e^{-2s}}{s} + \frac{1}{2}$$

$$(ii) X(s) = \int_{-1}^1 e^{-st} dt = \frac{e^{-s} - e^s}{s}$$

5.3 (a) Does not have a bilateral Laplace transform
(ROC for $x_+(t)$ is $\text{Re}\{s\} > 1$ and the
ROC for $x_-(t)$ is $\text{Re}\{s\} < 1$ and the overlap
is the null set)

$$(b) X(s) = \int_0^{\infty} \exp[bt] \exp[-st] dt = \frac{1}{s-b}$$

$$X_B(s) = \int \{ \exp[-bt] \}_{s \rightarrow -s} = \left\{ \frac{1}{s+b} \right\}_{s \rightarrow -s} = \frac{1}{-s+b}, \text{Re}\{s\} < b$$

(c) Does not have a bilateral Laplace transform

(d) Does not have a bilateral Laplace transform

$$(e) X_+(s) = \int_0^{\infty} \exp[-2t] \exp[-st] dt = \int_0^{\infty} \exp[-(2+s)t] dt = \frac{1}{s+2}$$

$$\text{noncausal part} = \int \{ \exp[-2t] \}_{s \rightarrow -s} = \frac{1}{-s+2}$$

$$X(s) = \frac{1}{s+2} + \frac{1}{-s+2} = \frac{4}{4-s^2}$$

(f) Does not have a bilateral Laplace transform.

$$(g) X_+^-(s) = 0$$

$$X(s) = \int \{ \cos at \}_{s \rightarrow -s} = \frac{-s}{s^2 + a^2}$$

$$(h) X_+(s) = 0$$

$$X(s) = \int \{ \sinh(-at) \}_{s \rightarrow -s} = \frac{2a}{a^2 - s^2}$$

$$5.4 \quad (a) \quad Y(s) = 3 \cdot \frac{1}{s} X\left(\frac{s}{3}\right) = \frac{\left(\frac{s}{3}\right)^3 + 2\left(\frac{s}{3}\right)^2 + 3\left(\frac{s}{3}\right) + 1}{\left(\frac{s}{3}\right)^4 + 2\left(\frac{s}{3}\right)^3 + 2\left(\frac{s}{3}\right)^2 + 2}$$

$$= \frac{3(s^3 + 6s^2 + 27s + 54)}{s^4 + 6s^3 + 18s^2 + 54s + 162}$$

$$(b) \quad Y(s) = - \frac{dX(s)}{ds} = - \frac{s^6 + 4s^5 + 11s^4 + 16s^3 + 8s^2 - 2}{(s^4 + 2s^3 + 2s^2 + 2s + 2)^2}$$

$$(c) \quad y(t) = (t-1)x(t-1) + x(t-1)$$

$$\therefore X(s) = (e^{-s} + 1)X(s) = \frac{(e^{-s} + 1)(s^3 + 2s^2 + 3s + 2)}{s^4 + 2s^3 + 2s^2 + 2s + 2}$$

$$(d) \quad Y(s) = sX(s) = \frac{s^4 + 2s^3 + 3s^2 + 2s}{s^4 + 2s^3 + 2s^2 + 2s + 2}$$

$$(e) \quad Y(s) = e^{-s}X(s) + sX(s)$$

$$= \frac{(e^{-s} + s)(s^3 + 2s^2 + 3s + 2)}{s^4 + 2s^3 + 2s^2 + 2s + 2}$$

$$(f) \quad Y(s) = \frac{1}{s}X(s) = \frac{s^3 + 2s^2 + 3s + 2}{s[s^4 + 2s^3 + 2s^2 + 2s + 2]}$$

$$5.5 \quad \mathcal{L}\{t^n\} = \int_0^{\infty} t^n \exp[-st] dt$$

$$= \frac{t^n \exp[-st]}{-s} \Big|_0^{\infty} - \frac{n}{(-s)} \int_0^{\infty} t^{n-1} \exp[-st] dt$$

$$= \frac{n}{s} \int_0^{\infty} t^{n-1} \exp[-st] dt$$

$$= \frac{n}{s} \left[\frac{t^{n-1} \exp[-st]}{-s} \Big|_0^{\infty} - \frac{n-1}{(-s)} \int_0^{\infty} t^{n-2} \exp[-st] dt \right]$$

$$= \frac{n(n-1)}{s^2} \int_0^{\infty} t^{n-2} \exp[-st] dt$$

$$= \frac{n(n-1)}{s^2} \left[\frac{t^{n-2} \exp[-st]}{-s} \Big|_0^{\infty} - \frac{n-2}{(-s)} \int_0^{\infty} t^{n-3} \exp[-st] dt \right]$$

$$= \frac{n(n-1)(n-2)}{s^3} \int_0^{\infty} t^{n-3} \exp[-st] dt = \dots$$

$$= \frac{n(n-1)(n-2)\dots 1}{s^n} \int_0^{\infty} \exp[-st] dt = \frac{n!}{s^{n+1}}$$

$$5.6 \quad \mathcal{L}\{t^a\} = \int_0^{\infty} t^a \exp[-st] dt$$

$$\Rightarrow \mathcal{L}\{t^a\} = \int_0^{\infty} t^a \exp[-st] dt = \int_0^{\infty} t^{(a+1-1)} \exp[-st] dt$$

$$\text{let } st = t' \quad dt = \frac{1}{s} dt'$$

$$\therefore \mathcal{L}\{t^a\} = \int_0^{\infty} \left(\frac{t'}{s}\right)^{(a+1-1)} \exp[-t'] \frac{1}{s} dt' = \int_0^{\infty} \frac{t'^{(a+1-1)}}{s^{a+1}} \exp[-t'] dt'$$

Compare with Gamma function

$$\mathcal{L}\{t^a\} = \frac{\Gamma(a+1)}{s^{a+1}} \quad a > 0$$

$$5.7 \quad \mathcal{L}\{t^n\} = \frac{\Gamma(n+1)}{s^{n+1}}$$

where

$$\begin{aligned} \Gamma(n+1) &= n \Gamma(n) = n \Gamma(n-1+1) = n(n-1) \Gamma(n-1) \\ &= n(n-1)(n-2)\dots 1 \Gamma(1) \end{aligned}$$

$$\Gamma(1) = \int_0^{\infty} \exp[-t] dt = 1$$

$$\therefore \Gamma(n+1) = n!$$

$$\text{i.e. } \mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}$$

$$5.8 \quad \mathcal{L}\{\cos \omega t\} = \int_0^{\infty} \cos \omega t \exp[-st] dt$$

$$\text{let } u = \cos \omega t \quad dv = \exp[-st] dt$$

$$\begin{aligned} \int_0^{\infty} \cos \omega t \exp[-st] dt &= \cos \omega t \left(-\frac{\exp[-st]}{s}\right) \Big|_0^{\infty} - \int_0^{\infty} \left(\frac{-\exp[-st]}{s}\right) (-\sin \omega t \cdot \omega) dt \\ &= \frac{1}{s} - \frac{\omega}{s} \int_0^{\infty} \exp[-st] \sin \omega t dt \end{aligned}$$

$$\text{let } u = \sin \omega t \quad dv = \exp[-st] dt$$

$$\begin{aligned} \int_0^{\infty} \cos \omega t \exp[-st] dt &= \frac{1}{s} - \frac{\omega}{s} \left[\sin \omega t \left(-\frac{\exp[-st]}{s}\right) \Big|_0^{\infty} \right. \\ &\quad \left. - \int_0^{\infty} \left(\frac{-\exp[-st]}{s}\right) \omega \cdot \cos \omega t \cdot dt \right] \end{aligned}$$

$$\Rightarrow \left(1 + \frac{\omega^2}{s^2}\right) \int_0^{\infty} \cos \omega t \exp[-st] dt = \frac{1}{s}$$

$$\text{or } \int_0^{\infty} \cos \omega t \exp[-st] dt = \frac{s}{s^2 + \omega^2}$$

$$\mathcal{L}\{\sin \omega t\} = \int_0^{\infty} \sin \omega t \exp[-st] dt$$

$$\text{let } u = \sin \omega t \quad dv = \exp[-st] dt$$

$$\begin{aligned} \int_0^{\infty} \sin \omega t \exp[-st] dt &= \sin \omega t \left(-\frac{\exp[-st]}{s} \right) \Big|_0^{\infty} - \int_0^{\infty} \left(-\frac{\exp[-st]}{s} \right) \omega \cos \omega t dt \\ &= \frac{\omega}{s} \int_0^{\infty} \exp[-st] \cos \omega t dt \end{aligned}$$

$$\text{Since } \int_0^{\infty} \exp[-st] \cos \omega t dt = \frac{s}{s^2 + \omega^2}$$

$$\therefore \int_0^{\infty} \sin \omega t \exp[-st] dt = \frac{\omega}{s} \left[\frac{s}{s^2 + \omega^2} \right] = \frac{\omega}{s^2 + \omega^2}$$

$$\begin{aligned} 5.9 \quad \cosh t &= \cos(jt) \quad \therefore \cosh(\omega_0 t) = \cos(j\omega_0 t) \\ \sinh t &= -j \sin(jt) \quad \therefore \sinh(\omega_0 t) = -j \sin(j\omega_0 t) \end{aligned}$$

$$\therefore \mathcal{L}[\cosh t u(t)] = \mathcal{L}[\cos(j\omega_0 t) u(t)]$$

$$= \frac{1}{j} \frac{\frac{j}{j}}{\left(\frac{j}{j}\right)^2 + \omega_0^2} = \frac{j}{j^2 - \omega_0^2}$$

$$\mathcal{L}[\sinh t u(t)] = -j \mathcal{L}[\sin(j\omega_0 t) u(t)]$$

$$= -j \times \frac{1}{j} \frac{\omega_0}{\left(\frac{j}{j}\right)^2 + \omega_0^2} = \frac{\omega_0}{j^2 - \omega_0^2}$$

$$5.10 \quad (a) \quad x(0^+) = \lim_{s \rightarrow \infty} \frac{s}{s+a} = 1$$

$$x(\infty) = \lim_{s \rightarrow 0} \frac{s}{s+a} = 0$$

$$(b) \quad x(0^+) = \lim_{s \rightarrow \infty} \frac{s}{(s+a)^2} = 0$$

Since $x(t) = \frac{1}{n-1!} t^{n-1} e^{-at} u(t)$, $x(\infty)$ does

$$(c) \quad x(0^+) = \lim_{s \rightarrow \infty} \frac{6}{s^2+25} = 0$$

Since $x(t)$ contains a sinusoidal term, $x(\infty)$ does not exist.

$$(d) \quad x(0^+) = \lim_{s \rightarrow \infty} \frac{s(s+2)}{s+3} = \infty$$

$$x(\infty) = \lim_{s \rightarrow 0} \frac{s(s+2)}{s+3} = 0$$

[Note that $x(t) = \delta(t) - e^{-3t} u(t)$]

$$(e) \quad x(0^+) = \lim_{s \rightarrow \infty} \frac{s(s^2+s+3)}{s^3+4s^2+2s+2} = 1$$

$$x(\infty) = \lim_{s \rightarrow 0} \frac{s(s^2+s+3)}{s^3+4s^2+2s+2} = 0$$

$$(f) \quad x(0^+) = \lim_{s \rightarrow \infty} \frac{s(s)}{s^2-2s-3} = 1$$

$$x(\infty) = \lim_{s \rightarrow 0} \frac{s^2}{s^2-2s-3} = 0$$

$$5.11 (a) \quad x(s) = \frac{s+2}{(s-1)(s+1)} = \frac{4/3}{s-2} + \frac{-1/3}{s+1}$$

$$x(t) = \left[\frac{4}{3} e^{2t} - \frac{1}{3} e^{-t} \right] u(t)$$

$$(b) \quad X(s) = \frac{s^2+8}{s(s^2+16)} = \frac{\frac{1}{2}}{s} + \frac{s}{s^2+16} - \frac{1}{8} \frac{4}{s^2+16}$$

$$x(t) = u(t) + \cos 2t u(t) - \frac{1}{8} \sin 2t u(t)$$

$$(c) \quad X(s) = \frac{s}{s^2+4} + \frac{s+1}{s^2+2s+2}$$

$$x(t) = \cos 2t u(t) + e^{-t} \cos t u(t)$$

$$(d) \quad X(s) = \frac{s^2}{s^2+3s+2} = 1 - \frac{4}{s+2} + \frac{1}{s+1}$$

$$x(t) = \delta(t) - 4e^{-2t} u(t) + e^{-t} u(t)$$

$$(e) \quad X(s) = \frac{s^2-s+1}{s^3-2s^2+s} = \frac{1}{s} + \frac{1}{(s-1)^2}$$

$$x(t) = u(t) + t e^t u(t)$$

$$(f) \quad \text{Let } X_1(s) = \frac{s+2}{s^2+2s+1} = \frac{1}{s+1} + \frac{1}{(s+1)^2}$$

$$\text{Then } x_1(t) = e^{-t} u(t) + t e^{-t} u(t)$$

$$\text{Since } X(s) = e^{-s} X_1(s), \text{ we have } x(t) = x_1(t-1)$$

$$\therefore x(t) = e^{-(t-1)} u(t-1) + (t-1) e^{-(t-1)} u(t-1)$$

$$(g) \quad X(s) = \frac{2s^2-6s+3}{s^2-3s+2} = 2 - \frac{1}{s-1} + \frac{1}{s-2}$$

$$\therefore x(t) = 2\delta(t) - e^t u(t) + e^{2t} u(t)$$

$$(h) \quad X(s) = \frac{2(s^2+2s+4)}{(s^2+4)^2} = 2 + \frac{4s}{(s^2+4)^2}$$

$$\therefore x(t) = 2\delta(t) + t \sin 2t u(t)$$

$$(i) \quad X(s) = \frac{2}{(2s+1)^3} = \frac{1}{8} \frac{2}{(s+\frac{1}{2})^3}$$

$$\therefore x(t) = \frac{1}{8} t^2 e^{-\frac{1}{2}t} u(t)$$

$$(j) \quad X(s) = \frac{(s^2+8)e^{-2s}}{s(s^2+16)} = \frac{\frac{1}{2}e^{-2s}}{s} + \frac{\frac{1}{2}s e^{-2s}}{s^2+16}$$

$$x(t) = \frac{1}{2} u(t-2) + \frac{1}{2} \cos[2(t-2)] u(t-2)$$

$$5.12 \quad (a) \quad Y(s) = \frac{1}{s-a} \cdot \frac{1}{s-b} = \frac{1}{a-b} \cdot \frac{1}{s-a} + \frac{1}{b-a} \cdot \frac{1}{s-b}$$

$$y(t) = \frac{1}{a-b} [e^{at} - e^{bt}] u(t)$$

$$(b) \quad Y(s) = \frac{1}{(s-a)^2}, \quad y(t) = t e^{at} u(t)$$

$$(c) \quad Y(s) = \left[\frac{e^s - e^{-s}}{s} \right] \frac{1}{s}$$

$$y(t) = (t+1)u(t+1) - (t-1)u(t-1) = \begin{cases} 0 & t \leq -1 \\ t+1 & -1 \leq t \leq 1 \\ 2 & t \geq 1 \end{cases}$$

$$(d) \quad Y(s) = \frac{1}{s^2} \cdot \frac{1}{s+a} = \frac{1/a}{s^2} - \frac{1/a^2}{s} + \frac{1/a^2}{s+a}$$

$$y(t) = \frac{1}{a} t u(t) - \frac{1}{a^2} u(t) + \frac{1}{a^2} e^{-at} u(t)$$

$$(e) \quad Y(s) = \frac{1}{s+b} \cdot \frac{1}{s} = \frac{-1/b}{s+b} + \frac{1/b}{s}$$

$$y(t) = \frac{1}{b} u(t) - \frac{1}{b} e^{-bt} u(t)$$

$$(f) \quad Y(s) = \frac{a}{s^2+a^2} \cdot \frac{s}{s^2+b^2} = \frac{\frac{a}{a^2-b^2} s}{s^2+b^2} - \frac{\frac{a}{a^2-b^2} s}{s^2+a^2}$$

$$\therefore y(t) = \frac{a}{a^2-b^2} [\cos(bt) + \cos(at)] u(t)$$

$$(g) \quad Y(s) = \frac{1}{s+2} \cdot \left[\frac{1-e^{-2s}}{s} \right] = (1-e^{-2s}) \left[\frac{1/2}{s} - \frac{1/2}{s+2} \right]$$

$$y(t) = \frac{1}{2} u(t) - \frac{1}{2} u(t-2) - \frac{1}{2} e^{-2t} u(t) + \frac{1}{2} e^{-2(t-2)} u(t-2)$$

$$(h) \quad Y(s) = \left[\frac{1}{s+2} + 1 \right] \frac{e^{-s}}{s} = \left[\frac{1/2}{s} - \frac{1/2}{s+2} + \frac{1}{s} \right] e^{-s}$$

$$y(t) = \frac{3}{2} u(t-1) - \frac{1}{2} e^{-2(t-1)} u(t-1)$$

$$5.13 \quad (a) \quad (i) \quad Y(s) = \frac{1}{s-a} \cdot \frac{1}{s-a}$$

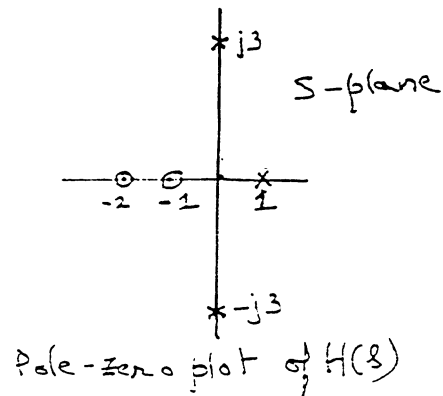
$$\therefore y(t) = e^{-at} u(t) * e^{-at} u(t) = t e^{-at} u(t)$$

$$(ii) \quad Y(s) = \frac{1}{(s-a)^2} \cdot \frac{1}{s-a}$$

$$\begin{aligned} \therefore y(t) &= t e^{-at} u(t) * e^{-at} u(t) \\ &= \int_{-\infty}^{\infty} \tau e^{-a\tau} u(\tau) e^{-a(t-\tau)} u(t-\tau) d\tau \\ &= \int_0^t \tau e^{-a\tau} d\tau = \frac{t^2}{2} e^{-at} u(t) \end{aligned}$$

$$(iii) \quad \mathcal{L}^{-1} \left[\frac{1}{(s-a)^n} \right] = \frac{t^{n-1}}{(n-1)!} e^{-at} u(t)$$

$$5.14 \quad (a) \quad H(s) = \frac{(s+1)(s+2)}{(s-1)(s^2+9)}$$



$$(b) \quad H(s) = \frac{3/5}{s-1} + \frac{2/5 s - 2}{s^2 + 9}$$

$$= \frac{3/5}{s-1} + \frac{2}{5} \frac{s}{s^2+9} - \frac{2}{3} \frac{3}{s^2+9}$$

$$\therefore h(t) = \frac{3}{5} e^t u(t) + \frac{2}{5} \cos 3t u(t) - \frac{2}{3} \sin 3t u(t)$$

and is real

$$\begin{aligned} (c) \quad H^*(s) &= \left[\int_0^{\infty} h(t) e^{st} dt \right]^* = \int_0^{\infty} h^*(t) e^{s^* t} dt \\ &= \int_0^{\infty} h(t) e^{s^* t} dt = H(s^*) \end{aligned}$$

Suppose $s = s_0$ is a zero of $H(s)$ and let

$$H(s) = (s - s_0) H_1(s)$$

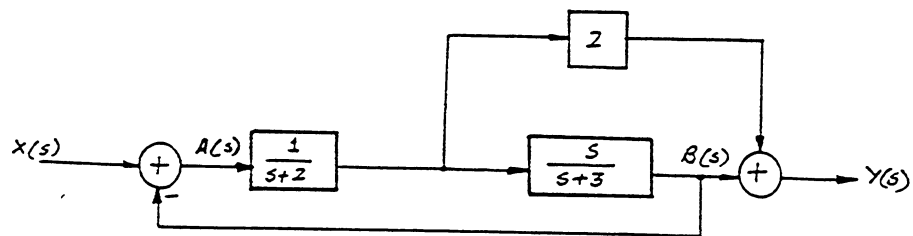
$$\text{Then } H(s_0) = H^*(s_0) = 0$$

But $H^*(s_0) = H(s_0^*)$ so that $H(s_0^*) = 0$
 i.e. s_0^* is a zero of $H(s)$

Similarly for a pole.

(d) $H(s)$ has zeros at $-1, -2$ and poles at $1, \pm j3$
 That is, poles and zeros occur in complex conjugate pairs.

5.15



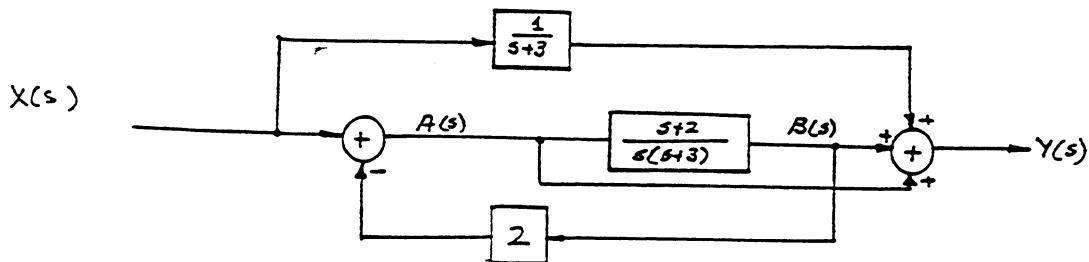
$$A(s) = X(s) - B(s)$$

$$B(s) = \frac{sA(s)}{(s+2)(s+3)}$$

$$A(s) = X(s) - \frac{sA(s)}{(s+2)(s+3)} \quad \text{or} \quad A(s) = \frac{X(s)}{1 + \frac{s}{(s+2)(s+3)}}$$

$$\begin{aligned} Y(s) &= B(s) + \frac{2A(s)}{s+2} = \left[\frac{s}{(s+2)(s+3)} + \frac{2}{s+2} \right] A(s) \\ &= \left[\frac{s}{(s+2)(s+3)} + \frac{2}{s+2} \right] \frac{X(s)}{1 + \frac{s}{(s+2)(s+3)}} \end{aligned}$$

$$\text{or } \frac{Y(s)}{X(s)} = \frac{s + 2(s+3)}{(s+2)(s+3) + s} = \frac{3(s+2)}{s^2 + 6s + 6}$$



$$A(s) = X(s) - 2B(s)$$

$$B(s) = \frac{s+2}{s(s+3)} A(s)$$

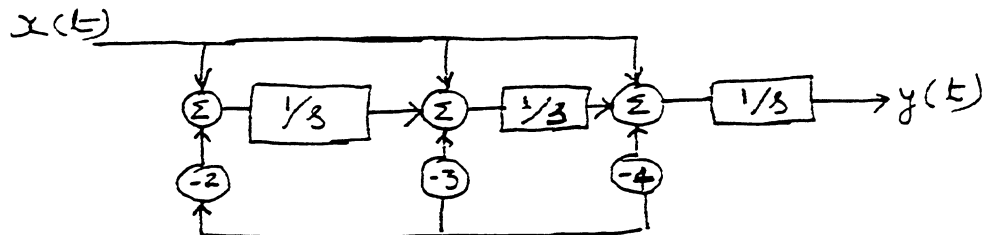
$$\therefore A(s) = X(s) - \frac{2(s+2)}{s(s+3)} A(s) \Rightarrow A(s) = \frac{X(s)}{1 + \frac{2(s+2)}{s(s+3)}}$$

$$B(s) = \frac{(s+2)A(s)}{s(s+3)} = \frac{s+2}{s(s+3)} \cdot \frac{X(s)}{1 + \frac{2(s+2)}{s(s+3)}} = \frac{(s+2)X(s)}{s^2 + 5s + 2}$$

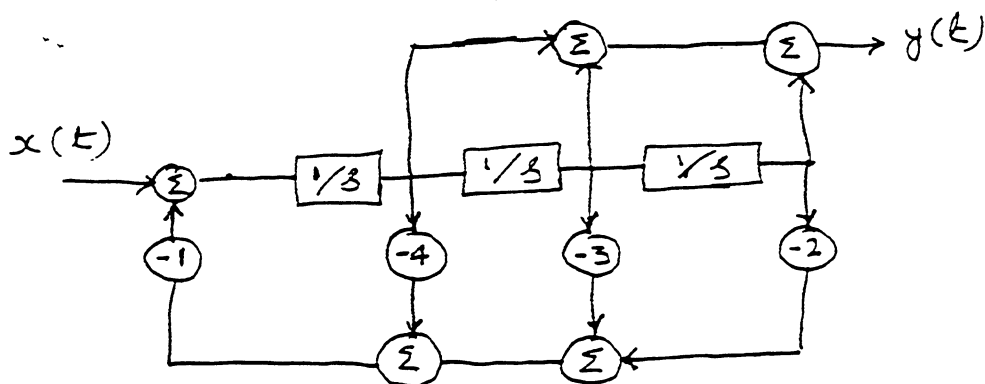
$$Y(s) = \frac{1}{s+3} X(s) + B(s) + A(s) \\ = \left[\frac{1}{s+3} + \frac{s+2}{s^2 + 5s + 2} + \frac{s(s+3)}{s^2 + 5s + 2} \right] X(s)$$

$$\text{or } \frac{Y(s)}{X(s)} = \frac{s^3 + 8s^2 + 19s + 8}{(s+3)(s^2 + 5s + 2)}$$

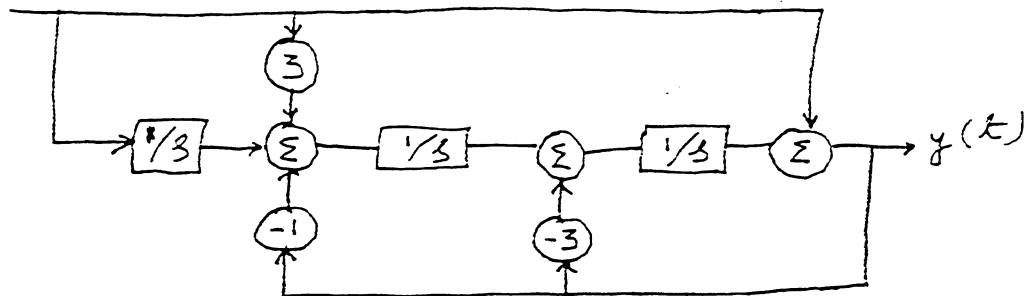
5.16 The first canonical form is



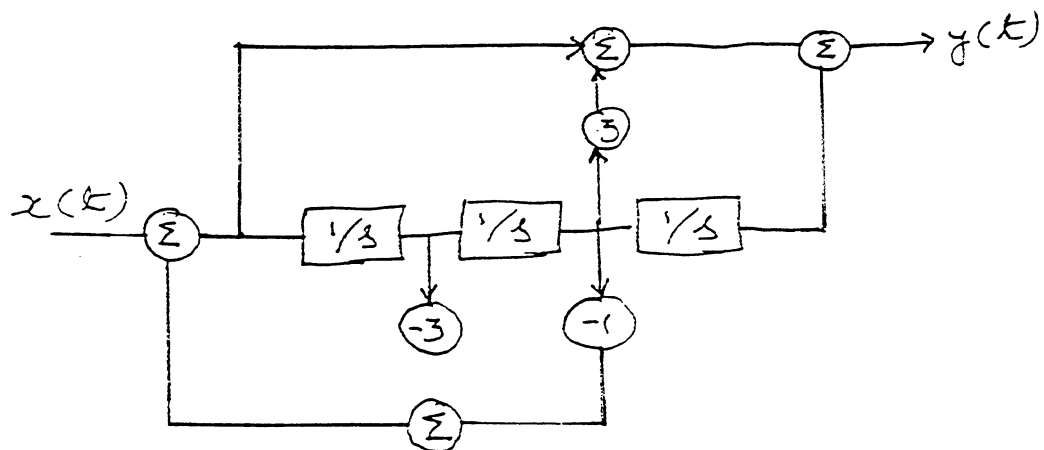
The second canonical form is



5.17 For the first form, we have



The second form is

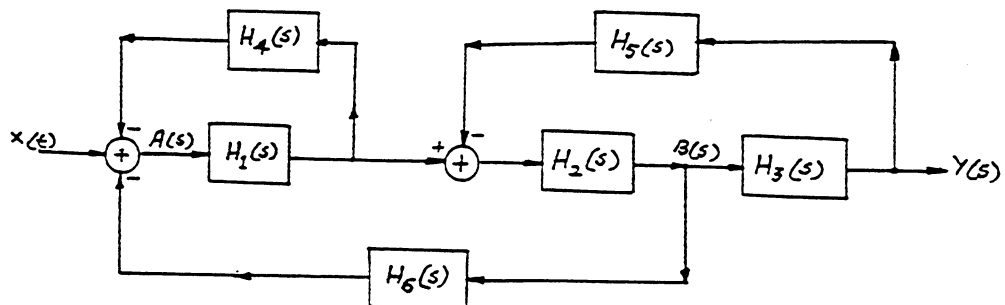


5.18 $2y''(t) + 3y'(t) + 4y(t) = 2x'(t) - x(t)$

$$(2s^2 + 3s + 4) Y(s) = (2s - 1) X(s)$$

$$\Rightarrow H(s) = \frac{Y(s)}{X(s)} = \frac{2s - 1}{2s^2 + 3s + 4}$$

5.19



$$(c) A = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}$$

$$\exp[At] = \mathcal{L}^{-1}[(sI-A)^{-1}] = \mathcal{L}^{-1} \begin{bmatrix} \frac{1}{s-1} & \frac{-2}{(s-1)^2} \\ 0 & \frac{1}{s-1} \end{bmatrix} = \begin{bmatrix} \exp[t] & -2t \exp[t] \\ 0 & \exp[t] \end{bmatrix}$$

$$(d) A = \begin{bmatrix} 3 & -2 \\ 1 & 1 \end{bmatrix}$$

$$\exp[At] = \mathcal{L}^{-1}[(sI-A)^{-1}] = \mathcal{L}^{-1} \begin{bmatrix} \frac{s-1}{(s-2)^2+1} & \frac{2}{(s-2)^2+1} \\ \frac{-1}{(s-2)^2+1} & \frac{s-3}{(s-2)^2+1} \end{bmatrix}$$

$$= \begin{bmatrix} \exp[2t] [\cos t + \sin t] & 2e^{2t} \sin t \\ -\exp[2t] \sin t & \exp[2t] [\cos t - \sin t] \end{bmatrix}$$

$$(e) A = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 1 \\ -1 & 0 & 0 \end{bmatrix}$$

$$\exp[At] = \mathcal{L}^{-1}[(sI-A)^{-1}] = \mathcal{L}^{-1} \begin{bmatrix} \frac{1}{s-1} & 0 & 0 \\ \frac{-(s+1)}{(s-1)^2 s} & \frac{1}{s-1} & \frac{1}{s(s-1)} \\ \frac{-1}{(s-1)s} & 0 & -\frac{1}{s} \end{bmatrix}$$

$$= \begin{bmatrix} \exp[t] & 0 & 0 \\ \exp[t] - 2t \exp[t] - u(t) & \exp[t] & \exp[t] - u(t) \\ -\exp[t] + u(t) & 0 & u(t) \end{bmatrix}$$

$$A(s) = X(s) - H_1(s) H_4(s) A(s) - H_6(s) B(s)$$

$$\text{or } [1 + H_1(s) H_4(s)] A(s) = X(s) - H_6(s) B(s) \quad \dots (1)$$

$$B(s) = H_1(s) H_2(s) A(s) - H_3(s) H_5(s) H_2(s) B(s)$$

$$\text{or } (1 + H_3 H_5) B(s) = H_1(s) H_2(s) A(s)$$

$$\text{or } A(s) = \frac{1 + H_3(s) H_5(s)}{H_1(s) H_2(s)}$$

$$\text{from (1)} \quad \frac{(1 + H_1(s) H_4(s)) (1 + H_3(s) H_5(s) H_2(s)) B(s)}{H_1(s) H_2(s)}$$

$$= X(s) - H_6(s) B(s)$$

$$\text{or } B(s) \left[\frac{(1 + H_1(s) H_4(s)) (1 + H_3(s) H_5(s) H_2(s)) + H_6(s) H_1(s) H_2(s)}{H_1(s) H_2(s)} \right]$$

$$= X(s)$$

$$H(s) = \frac{Y(s)}{X(s)} = \frac{H_3(s) B(s)}{X(s)}$$

$$= \frac{H_1(s) H_2(s) H_3(s)}{(1 + H_1(s) H_4(s)) (1 + H_3(s) H_5(s) H_2(s)) + H_6(s) H_1(s) H_2(s)}$$

$$5.20 (a) \quad y'(t) + 2y(t) = u(t) \quad y(0) = 1$$

$$sY(s) - y(0) + 2Y(s) = u(s)$$

$$\Rightarrow (s+2)Y(s) = 1 + \frac{1}{s} \quad \text{or } Y(s) = \frac{1}{s+2} + \frac{1}{s(s+2)}$$

$$= \frac{\frac{1}{2}}{s} + \frac{\frac{1}{2}}{s+2}$$

$$y(t) = \mathcal{L}^{-1}[Y(s)] = \frac{1}{2} u(t) + \frac{1}{2} \exp[-2t]$$

$$(b) \quad y'(t) + 2y(t) = \cos t \quad u(t) \quad y'(0) = 1$$

$$sY(s) - y'(0) + 2Y(s) = \frac{s}{s^2+1}$$

$$\Rightarrow Y(s) = \frac{1}{s+2} + \frac{s}{(s^2+1)(s+2)} = \frac{\frac{3}{5}}{s+2} + \frac{\frac{2}{5}(s+\frac{1}{2})}{s^2+1}$$

$$\therefore y(t) = \mathcal{L}^{-1}[Y(s)] = \frac{3}{5} \exp[-2t] + \frac{2}{5} \cos t + \frac{1}{5} \sin t$$

$$(c) \quad y'(t) + 2y(t) = \exp[-3t]u(t) \quad y(0) = 1$$

$$sY(s) - y(0) + 2Y(s) = \frac{1}{s+3}$$

$$\Rightarrow Y(s) = \frac{1}{s+3} + \frac{1}{(s+2)(s+3)}$$

$$y(t) = \mathcal{L}^{-1}[Y(s)] = \frac{3}{2} \exp[-2t] - \frac{1}{2} \exp[-3t]$$

$$(d) \quad y''(t) + 4y'(t) + 3y(t) = u(t) \quad y(0) = 2 \quad y'(0) = 1$$

$$s^2 Y(s) - sy(0) - y'(0) + 4sY(s) - 4y(0) + 3Y(s) = \frac{1}{s}$$

$$\Rightarrow (s^2 + 4s + 3)Y(s) = 2s + 9 + \frac{1}{s}$$

$$Y(s) = \frac{2s+9}{s^2+4s+3} + \frac{1}{s(s^2+4s+3)} = \frac{3}{s+1} + \frac{\frac{1}{3}}{s} + \frac{-\frac{4}{3}}{s+3}$$

$$y(t) = \mathcal{L}^{-1}[Y(s)] = 3\exp[-t] + \frac{1}{3}u[t] - \frac{4}{3}\exp[-3t]$$

$$(e) \quad y''(t) + 4y'(t) + 3y(t) = \exp[-3t]u(t)$$

$$s^2 Y(s) - sy'(0) - y(0) + 4sY(s) - 4y(0) + 3Y(s) = \frac{1}{s+3}$$

$$(s^2 + 4s + 3)Y(s) = 5y(0) + sy'(0) + \frac{1}{s+3}$$

$$Y(s) = \frac{5y(0) + sy'(0)}{s^2 + 4s + 3} + \frac{1}{(s+3)(s^2 + 4s + 3)}$$

$$= \frac{\frac{5}{2}y(0) - \frac{1}{2}y'(0)}{s+1} + \frac{-\frac{5}{2}y(0) + \frac{3}{2}y'(0)}{s+3} + \frac{\frac{1}{4}}{s+1} - \frac{1}{4} \frac{s+5}{(s+3)^2}$$

$$= \frac{\frac{5}{2}y(0) - \frac{1}{2}y'(0) + \frac{1}{4}}{s+1} + \frac{-\frac{5}{2}y(0) + \frac{3}{2}y'(0) - \frac{1}{4}}{s+3} - \frac{\frac{1}{2}}{(s+3)^2}$$

$$\Rightarrow y(t) = \mathcal{L}^{-1}[Y(s)] = \left(\frac{5}{2}y(0) - \frac{1}{2}y'(0) + \frac{1}{4} \right) \exp[-t]$$

$$+ \left(-\frac{5}{2}y(0) + \frac{3}{2}y'(0) - \frac{1}{4} \right) \exp[-3t] - \frac{1}{2} t \exp[-3t]$$

$$(f) y'''(t) + 3y''(t) + 2y'(t) - 6y(t) = \exp[-2]u(t),$$

$$y(0) = y'(0) = y''(0) = 0$$

$$s^3 Y(s) - s^2 y(0) - s y'(0) - y''(0) + 3[s^2 Y(s) - s y(0) - y'(0)]$$

$$+ 2[s Y(s) - y(0)] + 5 Y(s) = \frac{1}{s+2}$$

$$Y(s) = \frac{1}{(s^3 + 3s^2 + 2s - 6)(s+2)} = \frac{1}{(s-1)(s+2)(s^2 + 4s + 6)}$$

$$= \frac{\frac{1}{33}}{s-1} + \frac{-\frac{1}{6}}{s+2} + \frac{\frac{3}{22}s + \frac{2}{11}}{s^2 + 4s + 6}$$

$$y(t) = \frac{1}{33} \exp[t] - \frac{1}{6} \exp[-2t] + \frac{3}{22} \exp[-2t] (\cos \sqrt{2}t - \frac{1}{\sqrt{2}} \exp[-2t] \sin \sqrt{2}t)$$

$$5.21 (a) x(t) = \delta(t)$$

$$y'(t) + 5y(t) = x(t) + 2x'(t)$$

$$Y(s) = \frac{2s+1}{s+5} = 2 + \frac{-9}{s+5}$$

$$h(t) = \mathcal{L}^{-1}[Y(s)] = 2\delta(t) - 9\exp[-5t]$$

$$(b) y''(t) + 4y'(t) + 3y(t) = 2x(t) - 3x'(t) \quad x(t) = \delta(t)$$

$$H(s) = \frac{2-3s}{s^2 + 4s + 3} = \frac{\frac{5}{2}}{s+1} + \frac{-\frac{11}{2}}{s+3}$$

$$h(t) = \mathcal{L}^{-1}[H(s)] = \frac{5}{2} \exp[-t] - \frac{11}{2} \exp[-3t]$$

$$(c) y'''(t) + y'(t) - 2y(t) = x''(t) + x'(t) + 2x(t) \quad x(t) = \delta(t)$$

$$H(s) = \frac{s^2 + s + 2}{s^3 + s - 2} = \frac{1}{s-1}$$

$$h(t) = \mathcal{L}^{-1}[H(s)]$$

$$= \exp[t] u(t)$$

$$5.22 (a) \quad X(s) = \frac{2}{s+2}, \quad Y(s) = -\frac{1}{s^2} + \frac{1}{s+2} + \frac{1}{s+1} + \frac{1}{s}$$

$$= \frac{3s^3 + 5s^2 - s - 2}{s^2(s+1)(s+2)}$$

$$H(s) = \frac{Y(s)}{X(s)} = \frac{3s^3 + 5s^2 - s - 2}{2s^2(s+1)} = \frac{3}{2} + \frac{s^2 - \frac{5}{4}s - 1}{(s^2)(s+1)}$$

$$= \frac{3}{2} + \frac{\frac{5}{4}}{s+1} + \frac{-\frac{1}{4}}{s} - \frac{\frac{1}{4}}{s^2}$$

$$h(t) = \mathcal{L}^{-1}[H(s)] = \frac{3}{2} \delta(t) + \frac{5}{4} \exp[-t] u(t) - \frac{1}{4} u(t) - \frac{1}{4} t u(t)$$

$$(b) \quad X(s) = \frac{2}{s} \quad Y(s) = \frac{1}{s^2} - \frac{1}{s+2}$$

$$H(s) = \frac{Y(s)}{X(s)} = \frac{\frac{1}{s^2} - \frac{1}{s+2}}{\frac{2}{s}} = \frac{-s^2 + s + 2}{2s(s+2)} = -\frac{1}{2} + \frac{\frac{1}{2}}{s} + \frac{1}{s+2}$$

$$\Rightarrow h(t) = \mathcal{L}^{-1}[H(s)] = -\frac{1}{2} \delta(t) + \frac{1}{2} u(t) + \exp[-2t] u(t)$$

$$(c) \quad X(s) = \frac{1}{s+2}, \quad Y(s) = \frac{1}{s+1} + \frac{-3}{s+2}$$

$$H(s) = \frac{Y(s)}{X(s)} = \frac{\frac{1}{s+1} + \frac{-3}{s+2}}{\frac{1}{s+2}} = \frac{s+2}{s+1} - 3 = -2 + \frac{1}{s+1}$$

$$h(t) = \mathcal{L}^{-1}[H(s)] = -2\delta(t) + \exp[-t] u(t)$$

$$(d) \quad X(s) = \frac{1}{s^2} \quad Y(s) = \frac{2}{s^3} - \frac{2}{s+3}$$

$$H(s) = \frac{Y(s)}{X(s)} = \frac{\frac{2}{s^3} - \frac{2}{s+3}}{\frac{1}{s^2}} = \frac{2}{s} - \frac{2s^2}{s+3}$$

$$= \frac{2}{s} - 2s + 6 - \frac{18}{s+3}$$

$$\therefore h(t) = 2u(t) - 2\delta'(t) + 6\delta(t) - 18\exp[-3t] u(t)$$

$$(e) \quad X(s) = \frac{2}{s} \quad Y(s) = -\frac{1}{\sqrt{2}} \left(\frac{s+2}{(s+2)^2+16} + \frac{4}{(s+2)^2+16} \right)$$

$$= -\frac{1}{\sqrt{2}} \frac{s+6}{(s+2)^2+16}$$

$$H(s) = \frac{Y(s)}{X(s)} = \frac{-\frac{1}{\sqrt{2}} \left(\frac{s+6}{(s+2)^2+16} \right)}{\frac{2}{s}} = -\frac{1}{2\sqrt{2}} \frac{s^2+6s}{(s+2)^2+16}$$

$$= -\frac{1}{2\sqrt{2}} - \frac{1}{\sqrt{2}} \frac{s-10}{(s+2)^2+16}$$

$$h(t) = \mathcal{L}^{-1}[H(s)] = -\frac{1}{2\sqrt{2}} \delta(t) - \frac{1}{\sqrt{2}} \exp[-2t] \cos 4t$$

$$+ \frac{3}{\sqrt{2}} \exp[-2t] \sin 4t$$

$$(f) \quad X(s) = \frac{3}{s^2}, \quad Y(s) = -\frac{3}{\sqrt{2}} \frac{s+4}{(s+4)^2+16} + \frac{1}{\sqrt{2}} \frac{4}{(s+4)^2+16}$$

$$= \frac{1}{\sqrt{2}} \frac{-3s-8}{(s+4)^2+16}$$

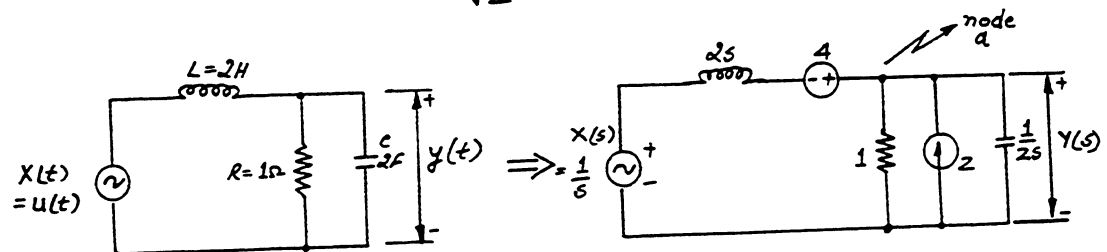
$$H(s) = \frac{Y(s)}{X(s)} = \frac{\frac{1}{\sqrt{2}} \frac{-3s-8}{(s+4)^2+16}}{\frac{3}{s^2}} = \frac{(-3s-8)s^2}{3\sqrt{2}[(s+4)^2+16]}$$

$$= \frac{-3s+16}{3\sqrt{2}} + \frac{-32s-512}{(s+4)^2+16}$$

$$h(t) = \mathcal{L}^{-1}[H(s)] = -\frac{1}{\sqrt{2}} \delta'(t) + \frac{16}{3\sqrt{2}} \delta(t) - \frac{32}{3\sqrt{2}} \exp[-4t] \cos 4t$$

$$- \frac{32}{\sqrt{2}} \exp[-4t] \sin 4t$$

5.23



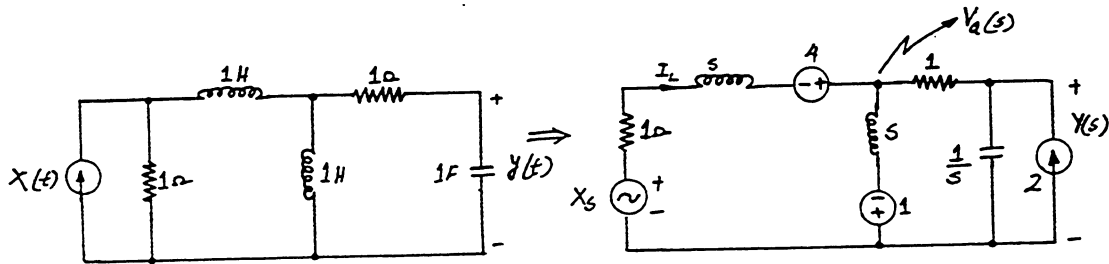
at node a, $\frac{\frac{1}{s} + 4 - Y(s)}{2s} + 2 - Y(s) - Y(s) 2s = 0$

$$\Rightarrow \frac{1}{s} + 4 - Y(s) + 4s = (1+2s)^2 Y(s)$$

$$\Rightarrow Y(s) = \frac{s^2 + s + \frac{1}{4}}{s(s^2 + \frac{1}{2}s + \frac{1}{4})} = \frac{1}{s} + \frac{\frac{1}{2}}{s^2 + \frac{1}{2}s + \frac{1}{4}}$$

$$y(t) = \mathcal{L}^{-1}[Y(s)] = u(t) + \frac{2}{\sqrt{5}} \exp[-\frac{1}{4}t] \sin \frac{\sqrt{5}}{4} t$$

5.24



use KVL

$$\frac{X + 4 - V_a}{1+s} = V_a - Y(s) + \frac{V_a + 1}{s}$$

$$\text{or } X + 4 = [V_a - Y(s)](1+s) + \frac{V_a + 1}{s} \quad (1)$$

$$V_a - Y(s) = -2 + sY(s)$$

$$V_a = -2 + (1+s)Y(s) \quad (2)$$

(2) substitute into (1)

$$\Rightarrow Y(s) = \frac{sX(s) + 2s^2 + 9s + 1}{(s+1)^3}$$

$$X(t) = u(t) \Rightarrow X(s) = \frac{1}{s}$$

$$Y(s) = \frac{2s^2 + 9s + 2}{(s+1)^3} = \frac{2(s+1)^2 + 5(s+1) - 5}{(s+1)^3}$$

$$= \frac{2}{s+1} + \frac{5}{(s+1)^2} - \frac{5}{(s+1)^3}$$

$$\therefore y(t) = (2 \exp[-t] + 5t \exp[-t] - \frac{5}{2} t^2 \exp[-t]) u(t)$$

$$5.25 \quad X(t) = \cos t \, u(t) \quad X(s) = \frac{s}{s^2+1}$$

$$\Rightarrow \frac{\frac{s}{s^2+1} + 4 - Y(s)}{2s} + 2 - Y(s) - 2s Y(s) = 0$$

$$\text{or } \frac{s}{s^2+1} + 4 - Y(s) + 4s = 2s(1+2s) Y(s)$$

$$\begin{aligned} \Rightarrow Y(s) &= \frac{\frac{s}{s^2+1} + 4s + 4}{4s^2 + 2s + 1} = \frac{s}{4(s^2+1) \left[\left(s + \frac{1}{4}\right)^2 + \left(\frac{\sqrt{3}}{4}\right)^2 \right]} \\ &\quad + \frac{s+1}{\left(s + \frac{1}{4}\right)^2 + \left(\frac{\sqrt{3}}{4}\right)^2} \\ &= \frac{-\frac{3}{13}s + \frac{2}{13}}{4(s^2+1)} + \frac{\frac{12}{13}s - \frac{2}{13} + s+1}{4 \left[\left(s + \frac{1}{4}\right)^2 + \left(\frac{\sqrt{3}}{4}\right)^2 \right]} \end{aligned}$$

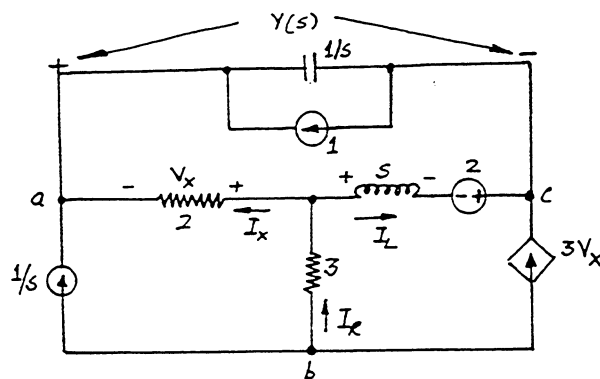
$$\begin{aligned} \Rightarrow y(t) &= -\frac{3}{\sqrt{2}} \cos t + \frac{2}{s^2} \sin t + \frac{25}{s^2} \exp\left[-\frac{1}{4}t\right] \cos \frac{\sqrt{3}}{4}t \\ &\quad + \frac{19}{52\sqrt{3}} \sin \frac{\sqrt{3}}{4}t \end{aligned}$$

$$5.26 \quad X(t) = \sin 2t \, u(t) \quad X(s) = \frac{2}{s^2+4}$$

$$\begin{aligned} \text{from 5.24} \quad Y(s) &= \frac{\frac{2s}{s^2+4} + 2s^2 + 9s + 1}{(s+1)^3} \\ &= \frac{2s}{(s^2+4)(s+1)^3} + \frac{2s^2 + 9s + 1}{(s+1)^3} \\ &= \frac{\frac{154}{37}s - \frac{56}{37}}{s^2+4} + \frac{\frac{154}{37}s^2 + \frac{1}{37}s + \frac{14}{37} + 2s^2 + 9s + 1}{(s+1)^3} \\ &= \frac{-\frac{154}{37}s - \frac{56}{37}}{s^2+4} + \frac{\frac{1}{37} [154(s+1)^2 + 124(s+1) - 227]}{(s+1)^3} \end{aligned}$$

$$\begin{aligned} \Rightarrow y(t) &= \left(-\frac{154}{37} \cos 2t - \frac{28}{37} \sin 2t + \frac{154}{37} \exp[-t] \right. \\ &\quad \left. + \frac{124}{37} t \exp[-t] - \frac{227}{74} t^2 \exp[-t] \right) u(t) \end{aligned}$$

5.27



$$\text{at node a: } \frac{1}{s} + \frac{V_x}{2} + 1 - sY = 0$$

$$\text{at node b: } \frac{1}{s} + 3V_x + I_R = 0$$

$$V_x + 2 + Y(s) - I_L s = 0$$

$$I_L + \frac{V_x}{2} = I_R \Rightarrow \frac{1}{s} + \frac{7}{2} V_x + I_L = 0$$

$$\Rightarrow \begin{cases} \frac{V_x}{2} - sY = -1 - \frac{1}{s} \\ V_x - sI_L + Y(s) = -2 \\ \frac{7}{2} V_x + I_L = -\frac{1}{s} \end{cases}$$

$$\begin{bmatrix} \frac{7}{2} & 1 & 0 \\ 1 & -s & 1 \\ \frac{1}{2} & 0 & -s \end{bmatrix} \begin{bmatrix} V_x \\ I_L \\ Y(s) \end{bmatrix} = \begin{bmatrix} -\frac{1}{s} \\ -2 \\ -1 - \frac{1}{s} \end{bmatrix}$$

$$\Rightarrow Y(s) = \frac{\frac{7}{2}s + 3 + \frac{1}{s}}{\frac{7}{2}s^2 + s + \frac{1}{2}} = \frac{-7s + 2}{7s^2 + 2s + 1} + \frac{2}{s}$$

$$y(t) = \mathcal{L}^{-1}[Y(s)] = 2u(t) - \exp\left[-\frac{1}{7}t\right] \left(\cos \frac{\sqrt{48}}{7}t u(t) + \frac{3}{\sqrt{48}} \exp\left[-\frac{1}{7}t\right] \sin \frac{\sqrt{48}}{7}t u(t) \right)$$

$$5.28 \text{ (a)} \quad Y(s) = \frac{H_1(s) H_2(s)}{1 + H_1(s) + H_2(s)} = \frac{29 \frac{1}{s(s+a)}}{1 + 29 \cdot \frac{1}{s(s+a)}} = \frac{29}{s^2 + as + 29}$$

(b) (i) $a=5$, $X(s) = \frac{1}{s}$

$$Y(s) = \frac{29}{s(s^2+5s+29)} = \frac{1}{s} + \frac{-s-5}{s^2+5s+29} = \frac{1}{s} + \frac{-(s+\frac{5}{2})-\frac{5}{2}}{(s+\frac{5}{2})^2 + \frac{91}{4}}$$

$$\therefore y(t) = u(t) - \exp[-\frac{5}{2}t] \left(\cos \frac{\sqrt{91}}{2} t - \frac{5}{\sqrt{91}} \exp[-\frac{5}{2}t] \sin \frac{\sqrt{91}}{2} t \right)$$

(ii) $a=3$

$$Y(s) = \frac{29}{s(s^2+3s+29)} = \frac{1}{s} + \frac{-s-3}{s^2+3s+29} = \frac{1}{s} + \frac{-(s+\frac{3}{2})-\frac{3}{2}}{(s+\frac{3}{2})^2 + (\frac{\sqrt{107}}{2})^2}$$

$$y(t) = u(t) - \exp[-\frac{3}{2}t] \left(\cos \frac{\sqrt{107}}{2} t - \frac{3}{\sqrt{107}} \exp[-\frac{3}{2}t] \sin \frac{\sqrt{107}}{2} t \right)$$

(iii) $a=1$

$$Y(s) = \frac{29}{s(s^2+s+29)} = \frac{1}{s} + \frac{-s-1}{s^2+s+29} = \frac{1}{s} + \frac{-(s+\frac{1}{2})-\frac{1}{2}}{(s+\frac{1}{2})^2 + \frac{115}{4}}$$

$$y(t) = u(t) - \exp[-\frac{1}{2}t] \left(\cos \frac{\sqrt{115}}{2} t - \frac{1}{\sqrt{115}} \exp[-\frac{1}{2}t] \sin \frac{\sqrt{115}}{2} t \right)$$

5.29 $\frac{Y(s)}{X(s)} = \frac{H_c(s) H(s)}{1 + H_c(s) H(s)}$

$$X(s) = \frac{A}{s} \quad H_c(s) = \frac{s+1}{s} \quad , \quad H(s) = \frac{1}{s+2}$$

$$(a) \quad Y(s) = \frac{H_c(s) H(s)}{1 + H_c(s) H(s)} = \frac{\frac{s+1}{s} \cdot \frac{1}{s+2} \cdot \frac{A}{s}}{1 + \frac{s+1}{s} \cdot \frac{1}{s+2}}$$

$$y(\infty) = \lim_{s \rightarrow 0} s Y(s) = \lim_{s \rightarrow 0} \frac{\frac{s+1}{s+2} \left(\frac{A}{s} \right)}{1 + \frac{s+1}{(s+2)s}} = \lim_{s \rightarrow 0} \frac{(s+1) A}{(s+2)s + s+1} = A$$

$$(b) \quad Y(s) = \frac{\frac{(s+1)A}{s}}{s(s+2)+s+1} = \frac{(s+1)A}{s(s^2+3s+1)} = \frac{A}{s} + \frac{(-s-2)A}{s^2+3s+1}$$

$$= \frac{A}{s} + \frac{\frac{-1-\sqrt{5}}{2\sqrt{5}}}{s - \frac{-3+\sqrt{5}}{2}} + \frac{\frac{1-\sqrt{5}}{2\sqrt{5}}}{s - \frac{-3-\sqrt{5}}{2}}$$

$$y(t) = A u(t) - \frac{(1+\sqrt{5})A}{2\sqrt{5}} \exp\left[\left(-\frac{3+\sqrt{5}}{2}\right)t\right] + \frac{(1-\sqrt{5})A}{2\sqrt{5}} \exp\left[\left(-\frac{3-\sqrt{5}}{2}\right)t\right]$$

$$\begin{aligned} \exp[t] = x(t) - y(t) &= \frac{(1+\sqrt{5})A}{2\sqrt{5}} \exp\left[-\frac{3+\sqrt{5}}{2}t\right] \\ &\quad - \frac{(1-\sqrt{5})A}{2\sqrt{5}} \exp\left[-\frac{3-\sqrt{5}}{2}t\right] \end{aligned}$$

(c) No, pole of $H(s)$ is in right-hand side

(d) No one of the zeros of $H_c(s)$ must be at the origin

$$\begin{aligned} 5.3a \text{ (a)} \quad A &= \begin{bmatrix} 0 & -1 \\ 1 & 2 \end{bmatrix} \quad \exp[At] = \mathcal{L}^{-1} [S I - A]^{-1} \\ &= \mathcal{L}^{-1} \begin{bmatrix} \frac{S-2}{(S-1)^2} & \frac{-1}{(S-1)^2} \\ \frac{1}{(S-1)^2} & \frac{S}{(S-1)^2} \end{bmatrix} \\ &= \begin{bmatrix} \exp[t] - t \exp[t] & -t \exp[t] \\ t \exp[t] & \exp[t] + t \exp[t] \end{bmatrix} \end{aligned}$$

$$(b) \quad A = \begin{bmatrix} 1 & -1 \\ 2 & 0 \end{bmatrix}$$

$$\exp[At] = \mathcal{L}^{-1} [S I - A]^{-1} = \mathcal{L}^{-1} \begin{bmatrix} \frac{S}{S^2-S+2} & \frac{-1}{S^2-S+2} \\ \frac{2}{S^2-S+2} & \frac{S-1}{S^2-S+2} \end{bmatrix}$$

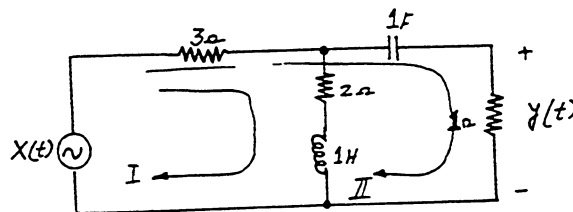
$$= \begin{bmatrix} \exp\left[\frac{t}{2}\right] \cos \frac{\sqrt{7}}{2} t + \frac{1}{\sqrt{7}} \exp\left[\frac{t}{2}\right] \sin \frac{\sqrt{7}}{2} t, & \frac{-2}{\sqrt{7}} \exp\left[\frac{t}{2}\right] \sin \frac{\sqrt{7}}{2} t \\ \frac{4}{\sqrt{7}} \exp\left[\frac{t}{2}\right] \sin \frac{\sqrt{7}}{2} t, & \exp\left[\frac{t}{2}\right] \cos \frac{\sqrt{7}}{2} t + \frac{3}{\sqrt{7}} \exp\left[\frac{t}{2}\right] \sin \frac{\sqrt{7}}{2} t \end{bmatrix}$$

$$(F) \quad A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 3 & 1 \\ 0 & -1 & 1 \end{bmatrix}$$

$$\exp[At] = \mathcal{L}^{-1}[(sI - A)^{-1}] = \mathcal{L}^{-1} \begin{bmatrix} \frac{1}{s-2} & \frac{1}{(s-2)^2} & \frac{1}{(s-2)^2} \\ 0 & \frac{s-1}{(s-2)^2} & \frac{1}{(s-2)^2} \\ 0 & \frac{1}{(s-2)^2} & \frac{s-3}{(s-2)^2} \end{bmatrix}$$

$$= \begin{bmatrix} \exp[2t] & t \exp[2t] & t \exp[2t] \\ 0 & \exp[2t] + t \exp[2t] & t \exp[2t] \\ 0 & t \exp[2t] & \exp[2t] - t \exp[2t] \end{bmatrix}$$

5.31



$$(a) \text{ loop I : } X(t) = (i_L + i_C) 3 + 2i_L + \frac{di_L}{dt} \quad \dots (1)$$

$$\text{loop II : } X(t) = (i_L + i_C) 3 + V_C + i_C \quad \dots (2)$$

$$\text{from (1) } X(t) = 5i_L + \frac{3dV_C}{dt} + \frac{di_L}{dt} \quad \dots (3)$$

$$\text{from (2) } X(t) = 3i_L + V_C + 4 \frac{dV_C}{dt} \quad \dots (4)$$

$$\text{from (3) \& (4) } \frac{di_L}{dt} = \frac{-11}{4} i_L + \frac{3}{4} V_C + \frac{X}{4}$$

$$\frac{dV_C}{dt} = -\frac{3}{4} i_L - \frac{1}{4} V_C + \frac{X}{4}$$

$$\Rightarrow \begin{bmatrix} \frac{di_L}{dt} \\ \frac{dV_C}{dt} \end{bmatrix} = \underbrace{\begin{bmatrix} \frac{-11}{4} & \frac{3}{4} \\ \frac{-3}{4} & \frac{-1}{4} \end{bmatrix}}_A \begin{bmatrix} i_L \\ V_C \end{bmatrix} + \underbrace{\begin{bmatrix} \frac{1}{4} \\ \frac{1}{4} \end{bmatrix}}_b$$

$$(b) \exp[At] = \mathcal{L}^{-1}[(sI - A)^{-1}]$$

$$= \mathcal{L}^{-1} \begin{bmatrix} \frac{s + 1/4}{(s + 1/2)(s + 5/2)} & \frac{3/4}{(s + 1/2)(s + 5/2)} \\ \frac{-3/4}{(s + 1/2)(s + 5/2)} & \frac{s + 1/4}{(s + 1/2)(s + 5/2)} \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{1}{8} \exp[-\frac{1}{2}t] + \frac{9}{8} \exp[-\frac{5}{2}t] & \frac{3}{8} \exp[-\frac{1}{2}t] - \frac{3}{8} \exp[-\frac{5}{2}t] \\ -\frac{3}{8} \exp[-\frac{1}{2}t] + \frac{3}{8} \exp[-\frac{5}{2}t] & \frac{9}{8} \exp[-\frac{1}{2}t] - \frac{1}{8} \exp[-\frac{5}{2}t] \end{bmatrix}$$

$$(c) \quad X(t) = u(t) \quad \dot{V}(0) = 0 \quad X(s) = \frac{1}{s}$$

$$\begin{bmatrix} I_L(s) \\ V_C(s) \end{bmatrix} = [sI - A]^{-1} b X(s)$$

$$= \begin{bmatrix} \frac{s + 1/4}{(s + 1/2)(s + 5/2)} & \frac{3/4}{(s + 1/2)(s + 5/2)} \\ \frac{-3/4}{(s + 1/2)(s + 5/2)} & \frac{s + 1/4}{(s + 1/2)(s + 5/2)} \end{bmatrix} \begin{bmatrix} 1/4 \\ 1/4 \end{bmatrix} \frac{1}{s}$$

$$V_C(s) = \frac{1}{4} \frac{(s+2) \frac{1}{s}}{(s + 1/2)(s + 5/2)}$$

$$y(t) = i_L = \frac{dV_C}{dt} \Rightarrow Y(s) = s V_C(s) - V_C(0) = s V_C(s)$$

$$y(t) = \mathcal{L}^{-1}[s V_C(s)] = \mathcal{L}^{-1}\left[\frac{1}{4} \frac{s+2}{(s + 1/2)(s + 5/2)}\right]$$

$$= \left(\frac{3}{16} \exp[-\frac{t}{2}] + \frac{1}{16} \exp[-\frac{5}{2}t] \right) u(t)$$

S-32

$$(a) \quad Y(s) = [sI - A]^{-1} Y(0^-)$$

$$= \begin{bmatrix} s+1 & 0 \\ 3 & s+2 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$= \frac{1}{(s+1)(s+2)} \begin{bmatrix} s+2 & 0 \\ -3 & s+1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1/(s+1) \\ -3/(s+1) + 2/(s+2) \end{bmatrix}$$

$$\therefore \underline{v}(t) = \begin{bmatrix} e^{-t} u(t) \\ -3e^{-t} u(t) + 2e^{-2t} u(t) \end{bmatrix}$$

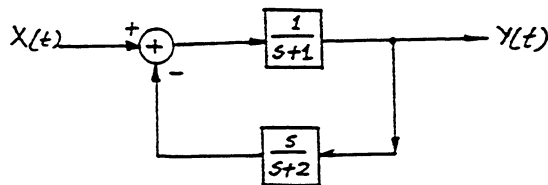
$$(b) \quad \underline{v}(s) = \begin{bmatrix} s-2 & 7 \\ -1 & s+3 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

$$= \frac{1}{s^2 + s + 1} \begin{bmatrix} s+3 & -7 \\ 1 & s-2 \end{bmatrix} \begin{bmatrix} 0 \\ -1 \end{bmatrix} = \begin{bmatrix} \frac{7}{s^2 + s + 1} \\ \frac{s+2}{s^2 + s + 1} \end{bmatrix}$$

$$= \begin{bmatrix} 14 \times \frac{\frac{1}{2}}{(s + \frac{1}{2})^2 + (\frac{\sqrt{3}}{2})^2} \\ - \frac{s + \frac{1}{2}}{(s + \frac{1}{2})^2 + (\frac{\sqrt{3}}{2})^2} + 5 \frac{\frac{1}{2}}{(s + \frac{1}{2})^2 + (\frac{\sqrt{3}}{2})^2} \end{bmatrix}$$

$$= \begin{bmatrix} 14 e^{-\frac{1}{2}t} \sin \frac{\sqrt{3}}{2} t u(t) \\ - e^{-\frac{1}{2}t} \cos \frac{\sqrt{3}}{2} t u(t) + 5 e^{-\frac{1}{2}t} \sin \frac{\sqrt{3}}{2} t u(t) \end{bmatrix}$$

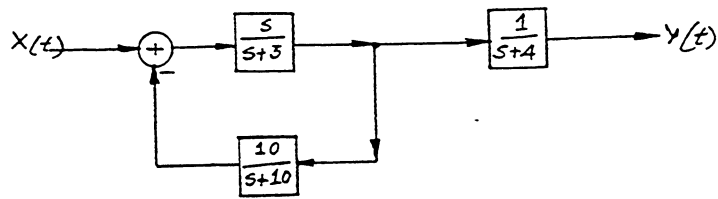
5.33 (a)



$$H(s) = \frac{Y(s)}{X(s)} = \frac{\frac{1}{s+1}}{1 + \frac{1}{s+1} \cdot \frac{s}{s+2}} = \frac{s+2}{s^2 + 4s + 2}$$

The two poles of $H(s)$, $s_1 = -3.414$, $s_2 = -0.5858$
 $\Rightarrow$ stable

(b)



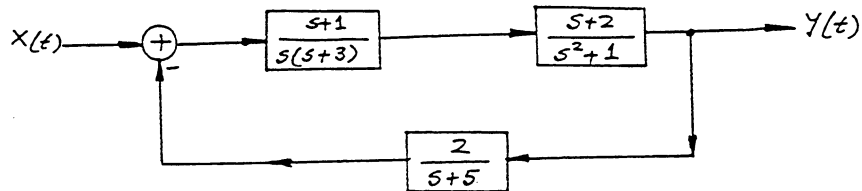
$$H(s) = \frac{Y(s)}{X(s)} = \frac{\frac{s}{s+3} \cdot \frac{1}{s+4}}{1 + \frac{s}{s+3} \cdot \frac{10}{s+10}} = \frac{s(s+10)}{s^3 + 27s^2 + 122s + 120}$$

the poles of $H(s)$

$$s_1 = -1.388, \quad s_2 = -21.6, \quad s_3 = -4$$

are all in the left-side plane $\Rightarrow$ stable

(c)

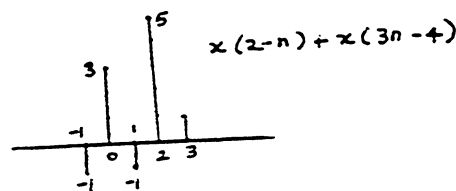
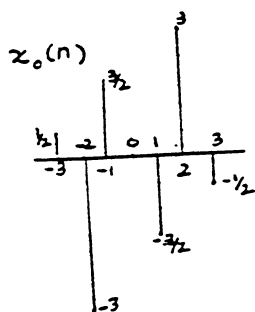
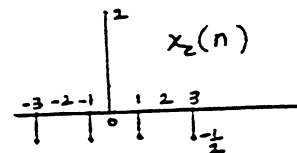
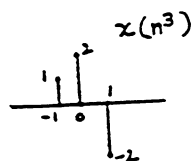
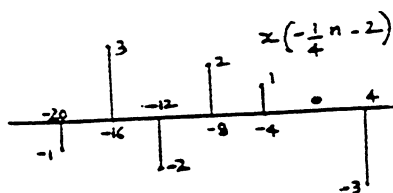
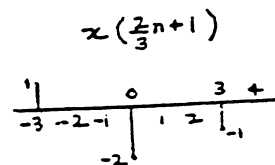
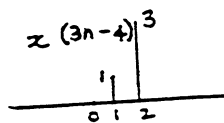
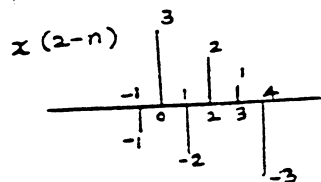


$$\begin{aligned} H(s) = \frac{Y(s)}{X(s)} &= \frac{\frac{s+1}{s(s+3)} \left(\frac{s+2}{s^2+1} \right)}{1 + \frac{(s+1)}{s(s+3)} \cdot \frac{(s+2)}{(s^2+1)} \cdot \frac{2}{(s+5)}} \\ &= \frac{(s+1)(s+5)(s+2)}{s^5 + 8s^4 + 16s^3 + 10s^2 + 21s + 4} \end{aligned}$$

Since there are two poles in the right-hand plane the system is unstable.

CHAPTER 6

6.1



6.2 (a) $x(2-n) = \{-1, -\frac{3}{2}, 1, 0, \frac{1}{3}, -1\}$

(b) $x(3n-4) = \{0, -1, 1\}$

(c)

$$x(\frac{2}{3}n+1) = \begin{cases} -1 & n = -2, 11 \\ 1/3 & n = 0 \\ 1 & n = 5 \\ -3/2 & n = 8 \\ 0 & \text{otherwise} \end{cases}$$

(d)

$$x(-\frac{n+8}{4}) = \begin{cases} -1 & n = -24, -4 \\ -3/2 & n = -20 \\ 1 & n = -16 \\ 1/3 & n = -8 \\ 0 & \text{otherwise} \end{cases}$$

$$(e) \quad x(n^3) = \begin{cases} -1 & n = -1 \\ \frac{1}{3} & n = 0 \\ 1 & n = 8 \\ -\frac{3}{2} & n = 27 \\ -1 & n = 64 \\ 0 & \text{otherwise} \end{cases}$$

$$(f) \quad x_e(n) = \frac{x(n) + x(-n)}{2} = \left\{ \frac{-1}{2}, -\frac{3}{4}, \frac{1}{2}, -\frac{1}{2}, \frac{2}{3}, \frac{1}{2}, -\frac{3}{4}, -\frac{1}{2} \right\}$$

$$(g) \quad x_o(n) = \frac{x(n) - x(-n)}{2} = \left\{ \frac{1}{2}, \frac{3}{4}, -\frac{1}{2}, -\frac{1}{2}, 0, \frac{1}{2}, \frac{1}{2}, -\frac{3}{4}, -\frac{1}{2} \right\}$$

$$(h) \quad x(2-n) + x(3n-4) = \left\{ -1, -\frac{3}{2}, 1, -1, \frac{4}{3}, -1 \right\}$$

6.3

$$(a) \quad \frac{\Omega_0}{2\pi} = \frac{\pi/4}{2\pi} = \frac{1}{8}; \quad N = 8$$

$$(b) \quad \frac{\Omega_{01}}{2\pi} = \frac{3\pi/4}{2\pi} = \frac{3}{8}; \quad N_1 = 8; \quad \frac{\Omega_{02}}{2\pi} = \frac{\pi/3}{2\pi} = \frac{1}{6}; \quad N_2 = 6$$

$x(n)$ is periodic with $N = 24$.

$$(b) \quad x(n) = \frac{1}{2} \cos\left(\frac{13\pi}{12} n\right) - \frac{1}{2} \cos\left(\frac{5\pi}{12} n\right)$$

so that

$$\frac{\Omega_{01}}{2\pi} = \frac{13\pi/12}{2\pi} = \frac{13}{24}; \quad N_1 = 24; \quad \frac{\Omega_{02}}{2\pi} = \frac{5\pi/12}{2\pi} = \frac{5}{24}; \quad N_2 = 24$$

$x(n)$ is periodic with $N = 24$.

(d) Not periodic

$$(e) \quad \frac{\Omega_0}{2\pi} = \frac{5\pi/6}{2\pi} = \frac{5}{12}; \quad N = 12$$

(f) $N_1 = 2, \quad N_2 = 3$ so that $N = 6$

$$(g) \frac{\Omega_{01}}{2\pi} = \frac{3\pi/4}{2\pi} = \frac{3}{8}; N_1 = 8; \frac{\Omega_{02}}{2\pi} = \frac{\pi/3}{2\pi} = \frac{1}{6}; N_2 = 6$$

so that $N = 24$.

$$6.4 \quad x(n) = x_0(nT) = 5 \cos 120 \left(\frac{\pi}{T} n - \frac{\pi}{3} \right)$$

$$\therefore \Omega_0 = 120 \pi T, \quad N = \frac{2\pi m}{120 \pi T} = \frac{Tm}{60}$$

Periodic for any integer T

$$6.5 \quad x(n) = 3 \sin 100 n \pi T + 4 \cos 120 n \pi T$$

$$N_1 = \frac{2\pi m}{100 \pi T} = \frac{m}{50T}, \quad N_2 = \frac{2\pi L}{120 \pi T} = \frac{L}{60T} \quad \text{for any integer } T.$$

$$N = pN_1 = qN_2 \quad \text{so that} \quad p = \frac{5qL}{cm}$$

Periodic for any integer T

6.6 Equality (i) for $x \neq 1$ and equalities (ii) and (iii) can be established by dividing the denominator into the numerator by long division

For $x = 1$, left side of equality (i) becomes

$$\sum_{n=0}^{N-1} (1) = N$$

6.7 (i) N.L., S.I., Memoryless, Causal

(ii) N.L., S.I., Memory, Causal

(iii) L., S.V., Memoryless, Causal

(iv) N.L., S.V., memoryless, Causal

(v) L., S.I., memory, Causal

(vi) L., S.I., memory, Causal

(vii) Linear, Memory, noncausal

$$\text{Let } x_1(n) = x(n-N), \text{ then } y_1(n) = \sum_{k=0}^{\infty} x_1(k) = \sum_{k=0}^{\infty} x(k-N) \\ = \sum_{k=-N}^{\infty} x(k)$$

$$y(n-N) = \sum_{k=0}^{\infty} x(k)$$

$$\therefore y_1(n) \neq y(n-N) \quad \therefore \text{Not S.I.}$$

$$(vii) \quad y(n) = \sum_{k=0}^n x(n) = (n+1)x(n)$$

L, S.V.,

$$(ix) \quad y(n) = \frac{1}{N} \sum_{k=0}^{N-1} x(n-k) \quad \text{memoryless}$$

L, S.I., Memory, Causal

(x) L, S.I., Memory, Noncausal

(xi) Nonlinear, memory, noncausal, S.I.

(xii) Linear, memoryless, Causal, S.V.

(xiii) Nonlinear, S.I., memoryless, Causal

6.8

$$(i) \quad y(n) = \sum_{k=-\infty}^n x(k) = \begin{cases} 2(-n-6) & -5 \leq n \leq -1 \\ 2(n-4) & 0 \leq n \leq 4 \\ 0 & \text{otherwise} \end{cases}$$

$$\begin{aligned} (ii) \quad y(n) &= \left(\frac{1}{2}\right)^n u(n) + \left(\frac{1}{2}\right)^{n-1} u(n-1) + \left(\frac{1}{2}\right)^n u(n) * \left(\frac{1}{3}\right)^n u(n) \\ &= \left(\frac{1}{2}\right)^n u(n) + \left(\frac{1}{2}\right)^{n-1} u(n-1) + 6 \left[\left(\frac{1}{2}\right)^{n+1} - \left(\frac{1}{3}\right)^{n+1} \right] u(n) \\ &= 4 \left(\frac{1}{2}\right)^n u(n) + \left(\frac{1}{2}\right)^{n-1} u(n-1) - 2 \left(\frac{1}{3}\right)^n u(n) \end{aligned}$$

$$(iii) \quad y(n) = \sum_{k=0}^n h(n) = \begin{cases} n+1 & 0 \leq n \leq 9 \\ 10 & n \geq 10 \end{cases}$$

$$\begin{aligned} (iv) \quad y(n) &= \left(\frac{1}{3}\right)^n u(n) + \left(\frac{1}{3}\right)^n * \left(\frac{1}{3}\right)^n u(n) \\ &= \left(\frac{1}{3}\right)^n u(n) + (n+1) \left(\frac{1}{3}\right)^n \end{aligned}$$

$$\begin{aligned} (v) \quad y(n) &= \left(\frac{1}{3}\right)^n u(n) + \left(\frac{1}{3}\right)^n * \left(\frac{1}{2}\right)^n u(n) \\ &= \left(\frac{1}{3}\right)^n u(n) + 3 \left(\frac{1}{2}\right)^n u(n) - 2 \left(\frac{1}{3}\right)^n \\ &= \left[3 \left(\frac{1}{2}\right)^n u(n) - 2 \left(\frac{1}{3}\right)^n \right] u(n) \end{aligned}$$

$$\begin{aligned}
 \text{(vi)} \quad y(n) &= \begin{cases} \sum_{k=0}^n k & 0 \leq n \leq 9 \\ \sum_{k=n-9}^n k & n \geq 10 \end{cases} \\
 &= \begin{cases} \frac{1}{2}n(n+1) & 0 \leq n \leq 9 \\ 10n-4 & n \geq 10 \end{cases}
 \end{aligned}$$

6.9(i)

n	-2	-1	0	1	2	3	4	5	
x(n)	1	-1/2	1/4	-1/8	1/16				x 1
x(n-1)		1	-1/2	1/4	-1/8	1/16			x -1
x(n-2)			1	-1/2	1/4	-1/8	1/16		x 1
x(n-3)				1	-1/2	1/4	-1/8	1/16	x -1

$$y(n) = \left\{ 1, -\frac{3}{2}, \frac{7}{4}, -\frac{15}{8}, \frac{15}{16}, -\frac{7}{16}, \frac{3}{16}, -\frac{1}{16} \right\}$$

(ii)

n	0	1	2	3	4	5	6	7	8
2x x(n)	1	2	3	0	-1				
-1x x(n-1)		1	2	3	0	-1			
3x x(n-2)			1	2	3	0	-1		
1x x(n-3)				1	2	3	0	-1	
-2x x(n-4)					1	2	3	0	-1

$$y(n) = \{ 2, 3, 7, 4, 7, 0, -9, -1, 2 \}$$

(iii)

n	-2	-1	0	1	2	3	4	5
2x x(n)	3	1/2	-1/4	1	4			
-1x x(n-1)		3	1/2	-1/4	1	4		
1/2x x(n-2)			3	1/2	-1/4	1	4	
-1/2x x(n-3)				3	1/2	-1/4	1	4

$$y(n) = \left\{ 6, -2, \frac{1}{2}, 1, \frac{65}{3}, -3\frac{3}{5}, \frac{3}{2}, -2 \right\}$$

(iv)	n	0	1	2	3	4	5	6	7	8
	$x(n)$	-1	$1/2$	$3/4$	$-1/5$	1				
	$x(n-1)$		-1	$1/2$	$3/4$	$-1/5$	1			
	$x(n-2)$			-1	$1/2$	$3/4$	$-1/5$	1		
	$x(n-3)$				-1	$1/2$	$3/4$	$-1/5$	1	
	$x(n-4)$					-1	$1/2$	$3/4$	$-1/5$	1

$$y(n) = \left\{ -1, -1/2, 1/4, 1/20, 21/20, 41/20, 31/20, 4/5, 1 \right\}$$

6.10 (a) $h(n) = [h_1(n) * h_2(n) - h_4(n)] * h_3(n)$

$$\begin{aligned}
 &= \left[(n+1) \left(\frac{1}{3}\right)^n u(n) - \left(\frac{1}{2}\right)^n u(n) \right] * u(n) \\
 &= \sum_{k=0}^n (k+1) \left(\frac{1}{3}\right)^k - \sum_{k=0}^n \left(\frac{1}{2}\right)^k \\
 &= -\frac{1}{2} n \left(\frac{1}{3}\right)^n u(n) - \frac{5}{4} \left(\frac{1}{3}\right)^n u(n) + \frac{9}{4} u(n) - 2u(n) + \left(\frac{1}{2}\right)^n u(n) \\
 &= -\frac{1}{2} n \left(\frac{1}{3}\right)^n u(n) - \frac{5}{4} \left(\frac{1}{3}\right)^n u(n) + \frac{1}{4} u(n) + \left(\frac{1}{2}\right)^n u(n)
 \end{aligned}$$

(b) $s(n) = \sum_{k=0}^n h(k) = \sum_{k=0}^n \left[-\frac{1}{2} k \left(\frac{1}{3}\right)^k - \frac{5}{4} \left(\frac{1}{3}\right)^k + \frac{1}{4} + \left(\frac{1}{2}\right)^k \right]$

$$= \left[-\frac{1}{6} \left(\frac{1}{3}\right)^n + \frac{5}{24} \left(\frac{1}{3}\right)^n - \left(\frac{1}{2}\right)^n + \frac{1}{4} n + \frac{9}{8} \right] u(n)$$

6.11 (a) $h(n) = \left[\left(\frac{1}{2}\right)^n u(n) * \delta(n) - \left(\frac{1}{3}\right)^n u(n) \right] * \left(\frac{1}{3}\right)^n u(n)$

$$\begin{aligned}
 &= \left(\frac{1}{2}\right)^n u(n) * \left(\frac{1}{3}\right)^n u(n) - \left(\frac{1}{3}\right)^n u(n) * \left(\frac{1}{3}\right)^n u(n) \\
 &= 6 \left[\left(\frac{1}{2}\right)^{n+1} - \left(\frac{1}{3}\right)^{n+1} \right] u(n) - (n+1) \left(\frac{1}{3}\right)^n u(n) \\
 &= 3 \left(\frac{1}{2}\right)^n u(n) - 3 \left(\frac{1}{3}\right)^n u(n) - n \left(\frac{1}{3}\right)^n u(n)
 \end{aligned}$$

(b) $s(n) = \sum_{k=0}^n h(k) = \sum_{k=0}^n \left[3 \left(\frac{1}{2}\right)^k - 3 \left(\frac{1}{3}\right)^k - k \left(\frac{1}{3}\right)^k \right]$

$$= \left[\frac{3}{4} - 3 \left(\frac{1}{2}\right)^n + \frac{9}{4} \left(\frac{1}{3}\right)^n - \frac{n}{2} \left(\frac{1}{3}\right)^n \right] u(n)$$

$$\begin{aligned}
6.12 \quad \sum_{k=n_0}^{N-1+n_0} x_1(k) x_2(n-k) &= \sum_{k=n_0}^{N-1} x_1(k) x_2(n-k) + \sum_{k=0}^{N-1+n_0} x_1(k) x_2(n-k) \\
&= \sum_{m=n_0+N}^{N-1} x_1(m-N) x_2(n-m+N) + \sum_{k=0}^{N-1+n_0} x_1(k) x_2(n-k) \\
&= \sum_{k=n_0+N}^{N-1} x_1(k) x_2(n-k) + \sum_{k=0}^{N-1+n_0} x_1(k) x_2(n-k) = \sum_{k=0}^{N-1} x_1(k) x_2(n-k)
\end{aligned}$$

- 6.13 (a) Do a circular shift of $x(n)$ so that the origin is at the left, to get

$$x(n) = \left\{ \frac{1}{4}, -\frac{1}{8}, \frac{1}{16}, 1, -\frac{1}{2} \right\}$$

Zero-pad $h(n)$ to get $h(n) = \{1, -1, 1, -1, 0\}$

Form the table

n	0	1	2	3	4	
$h(n)$	1	-1	1	-1	0	$x \frac{1}{4}$
$h(n-1)$	0	1	-1	1	-1	$x - \frac{1}{8}$
$h(n-2)$	-1	0	1	-1	1	$x \frac{1}{16}$
$h(n-2)$	1	-1	0	1	-1	$x 1$
$h(n-3)$	-1	1	-1	0	1	$x - \frac{1}{2}$
$y_p(n)$	$= \left\{ \frac{27}{16}, -\frac{15}{8}, \frac{15}{16}, \frac{9}{16}, -\frac{21}{16} \right\}$					

and is $y_i(n)$ of Problem 6.9(a) with the last three points aliased into the first 3 and the result circularly shifted by 2 points.

(b)

n	0	1	2	3	4	
$h(n)$	2	-1	3	1	-2	$x1$
$h(n-1)$	-2	2	-1	3	1	$x2$
$h(n-2)$	1	-2	2	-1	3	$x3$
$h(n-2)$	3	1	-2	2	-1	$x0$
$h(n-3)$	-1	3	1	-2	2	$x-1$
$y_p(n)$	$= \{2, -6, 6, 6, .7\}$					

and is $y_l(n)$ of Problem 6.9(b) with the last four points aliased into the first 4.

(c) Do a circular shift of $x(n)$ to get

$$x(n) = \left\{ -\frac{1}{4}, 1, 4, 1, 3, \frac{1}{2} \right\}$$

$$\text{Zero-pad } h(n) \text{ to get } h(n) = \left\{ 2, -1, \frac{1}{2}, -\frac{1}{2}, 0 \right\}$$

Form the table

n	0	1	2	3	4	
$h(n)$	2	-1	$\frac{1}{2}$	$-\frac{1}{2}$	0	$x-\frac{1}{4}$
$h(n-1)$	0	2	-1	$\frac{1}{2}$	$-\frac{1}{2}$	$x-\frac{1}{8}$
$h(n-2)$	$-\frac{1}{2}$	0	2	-1	$\frac{1}{2}$	$x\frac{1}{16}$
$h(n-2)$	$\frac{1}{2}$	$-\frac{1}{2}$	0	2	-1	$x1$
$h(n-3)$	-1	$\frac{1}{2}$	$-\frac{1}{2}$	0	2	$x-\frac{1}{2}$
$y_p(n)$	$= \left\{ -\frac{3}{2}, 1, \frac{13}{8}, \frac{21}{8}, -\frac{1}{2} \right\}$					

and is $y_l(n)$ of Problem 6.9(c) with the last three points aliased into the first 3 and the result circularly shifted left by 2 points.

(d)

n	0	1	2	3	4	
$h(n)$	1	1	1	1	1	x^{-1}
$h(n-1)$	1	1	1	1	1	$x^{-\frac{1}{2}}$
$h(n-2)$	1	1	1	1	1	$x^{-\frac{3}{4}}$
$h(n-3)$	1	1	1	1	1	$x^{-\frac{1}{5}}$
$h(n-4)$	1	1	1	1	1	x^{-1}

$$y_p(n) = \left\{ \frac{21}{20}, \frac{21}{20}, \frac{21}{20}, \frac{21}{20}, \frac{21}{20} \right\}$$

and is $y_l(n)$ of Problem 6.9(d) with the last three points aliased into the first 3.

$$6.14(a) \quad y(nT) + 2 \frac{y(nT) - y[(n-1)T]}{T} = x(nT)$$

$$(1 + \frac{2}{T}) y(nT) - \frac{2}{T} y[(n-1)T] = x(nT)$$

$$(b) \quad y(nT) + 2 \frac{y[(n+1)T] - y(nT)}{T} = x(nT)$$

$$\frac{2}{T} y[(n+1)T] + (1 - \frac{2}{T}) y(nT) = x(nT)$$

6.15/6.16

The tables below give selected values of the true value of the integrated $y(t)$, and the approximations using the backward difference $y_B(nT)$, the forward difference, $y_F(nT)$ and the trapezoidal approximation, $y_T(nT)$, computed using $T = 0.002$ and $T = 0.001$

$T = 0.002$	t	$y(t)$	$y_B(t)$	$y_F(t)$	$y_T(t)$
	0	0	0	0	0
	0.01	0.01	0.01	0.01	0.09
	0.05	0.05	0.05	0.05	0.05
	0.1	0.1	0.1	0.1	0.1
	0.25	0.25	0.25	0.25	0.25
	0.5	0.5	0.5	0.5	0.5
	0.75	0.75	0.75	0.75	0.75
	1	1	1	1	1
	1.01	1.00995	1.0	1.00994	1.00895
	1.1	1.095	1.0949	1.0949	1.094
	1.2	1.18	1.1798	1.1798	1.179
	1.5	1.375	1.3745	1.3745	1.374
	1.6	1.42	1.4194	1.4194	1.419
	1.7	1.455	1.4543	1.4543	1.454
	1.8	1.48	1.4792	1.4792	1.479
	1.9	1.495	1.4941	1.4941	1.495
	2.0	1.5	1.499	1.499	1.499
	3.0	1.5	1.499	1.499	1.499

$T = 0.001$	t	$y(t)$	$y_B(t)$	$y_F(t)$	$y_T(t)$
	0	0	0	0	0
	0.01	0.01	0.01	0.01	0.01
	0.05	0.05	0.05	0.05	0.05
	0.9	0.9	0.9	0.9	0.9
	1	1	1	1	1
	1.01	1.00995	1.009945	1.009946	1.00995
	1.1	1.095	1.09495	1.094951	1.095
	1.2	1.18	1.1799	1.179901	1.179999
	1.5	1.375	1.37475	1.374751	1.374999
	1.6	1.42	1.4197	1.419701	1.419999
	1.7	1.455	1.45465	1.454651	1.454999
	1.8	1.48	1.4796	1.479601	1.47999
	1.9	1.495	1.49455	1.494551	1.494999
	2.0	1.5	1.4995	1.4995	1.499999
	3.0	1.5	1.4995	1.4995	1.49999

It can be seen that all three methods provide good approximations to the integral in this example. The approximations get better as T becomes smaller.

6.17 (a) $y(n) = -y(n-1) - \frac{1}{4}y(n-2) + 1$

$$y(0) = -y(-1) - \frac{1}{4}y(-2) + 1 = 0 - 0.25 + 1 = 0.75$$

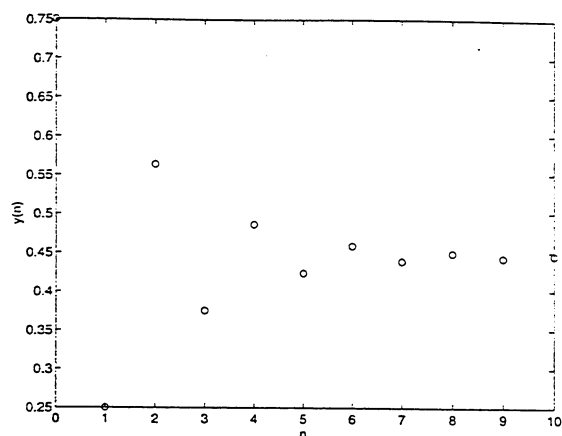
$$y(1) = -y(0) - \frac{1}{4}y(-1) + 1 = -0.75 + 1 = 0.25$$

$$y(2) = -y(1) - \frac{1}{4}y(0) + 1 = 0.5625$$

$$y(3) = -y(2) - \frac{1}{4}y(1) + 1 = 0.375$$

$$y(4) = -y(3) - \frac{1}{4}y(2) + 1 = 0.4844$$

$$y(5) = -y(4) - \frac{1}{4}y(3) + 1 = 0.4219$$



(b) $y(n) = -\frac{3}{4}y(n-1) - \frac{1}{8}y(n-2) + (\frac{1}{3})^n$

$$y(0) = -\frac{3}{4}y(-1) - \frac{1}{8}y(-2) + 1 = 1.75$$

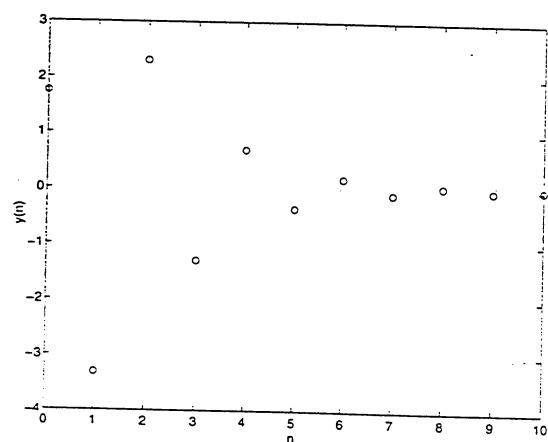
$$y(1) = -\frac{3}{4}y(0) - \frac{1}{8}y(-1) + \frac{1}{3} = -3.3125$$

$$y(2) = -\frac{3}{4}y(1) - \frac{1}{8}y(0) + \frac{1}{9} = 2.3027$$

$$y(3) = -\frac{3}{4}y(2) - \frac{1}{8}y(1) + \frac{1}{27} = -1.3006$$

$$y(4) = -\frac{3}{4}y(3) - \frac{1}{8}y(2) + \frac{1}{81} = 0.6917$$

$$y(5) = -\frac{3}{4}y(4) - \frac{1}{8}y(3) + \frac{1}{243} = -0.3548$$



(c) $y(n) = -\frac{3}{4}y(n-1) - \frac{1}{8}y(n-2) + (\frac{1}{2})^n$

$$y(0) = -\frac{3}{4}y(-1) - \frac{1}{8}y(-2) + 1 = 1$$

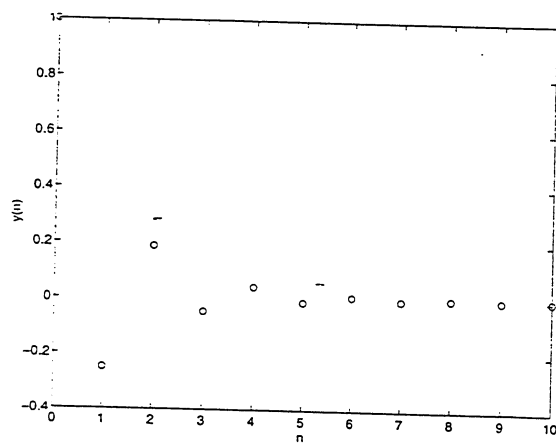
$$y(1) = -\frac{3}{4}y(0) - \frac{1}{8}y(-1) + \frac{1}{2} = -0.25$$

$$y(2) = -\frac{3}{4}y(1) - \frac{1}{8}y(0) + \frac{1}{4} = 0.1875$$

$$y(3) = -\frac{3}{4}y(2) - \frac{1}{8}y(1) + \frac{1}{8} = -0.0469$$

$$y(4) = -\frac{3}{4}y(3) - \frac{1}{8}y(2) + \frac{1}{16} = 0.043$$

$$y(5) = -\frac{3}{4}y(4) - \frac{1}{8}y(3) + \frac{1}{32} = -0.0107$$



$$(d) \quad y(n+1) = -\frac{1}{2}y(n-1) + x(n) - \frac{1}{2}x(n-1)$$

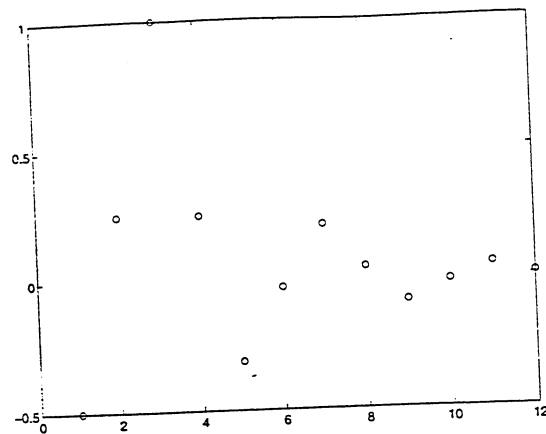
$$y(1) = -\frac{1}{2}y(0) + \frac{1}{4} - \frac{1}{2} = -0.5$$

$$y(2) = -\frac{1}{2}y(1) + \frac{1}{8} - \frac{1}{4} = 0.25$$

$$y(3) = -\frac{1}{2}y(2) + \frac{1}{16} - \frac{1}{8} = 1$$

$$y(4) = -\frac{1}{2}y(3) + \frac{1}{32} - \frac{1}{16} = -0.3125$$

$$y(5) = -\frac{1}{2}y(4) + \frac{1}{64} - \frac{1}{32} = -0.0312$$



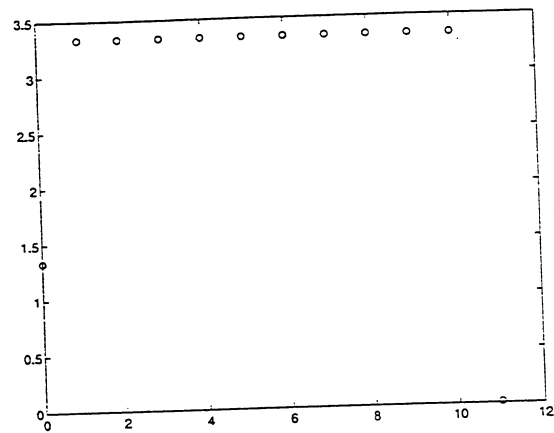
$$(e) \quad y(n) = x(n) + \frac{1}{3}x(n-1) + 2x(n-2)$$

$$y(0) = x(0) = 1$$

$$y(1) = 1 + \frac{1}{3} = 1\frac{1}{3}$$

$$y(2) = 1 + \frac{1}{3} + 2 = 3\frac{1}{3}$$

$$y(3) = y(4) = \dots = y(2) = 3\frac{1}{3}$$



6.18 (i) Characteristic equation is $z^2 - z + \frac{1}{2} = 0$ with roots

$$z = \frac{1}{2} \pm j\frac{1}{2} = e^{\pm j\frac{\pi}{4}} \text{ so that } y_h(n) = A_1 e^{j\frac{\pi}{4}n} + A_2 e^{-j\frac{\pi}{4}n}$$

Substituting the initial conditions and solving for A_1 and A_2 gives

$$y_h(n) = e^{j\frac{\pi}{4}(n+3)} + e^{-j\frac{\pi}{4}(n+3)} = 2\cos(\frac{\pi}{4}(n+3))u(n)$$

(ii) Characteristic equation is $z^2 - \frac{1}{6}z - \frac{1}{6} = 0$ with roots

$$z = -\frac{1}{3}, \frac{1}{2} \text{ so that } y_h(n) = A_1(-\frac{1}{3})^n + A_2(\frac{1}{2})^n$$

Solving for A_1 and A_2 gives

$$y_h(n) = \frac{1}{30}(-\frac{1}{3})^n u(n) + \frac{1}{10}(\frac{1}{2})^n u(n)$$

(iii) Characteristic equation is $z^2 - z + \frac{1}{4} = 0$ with roots

$$z = \frac{1}{2}, \frac{1}{2} \text{ so that } y_h(n) = A_1\left(\frac{1}{2}\right)^n + A_2 n\left(\frac{1}{2}\right)^n$$

Solving for A_1 and A_2 gives

$$y_h(n) = \frac{3}{4}\left(-\frac{1}{2}\right)^n u(n) - \frac{1}{4} n\left(-\frac{1}{2}\right)^n u(n)$$

(iv) Characteristic equation is $z^2 - \frac{3}{2}z + \frac{1}{2} = 0$ with roots

$$z = 1, \frac{1}{2} \text{ so that } y_h(n) = A_1 + A_2\left(\frac{1}{2}\right)^n.$$

Solving for A_1 and A_2 gives

$$y_h(n) = 2u(n) - \frac{1}{2}\left(\frac{1}{2}\right)^n u(n)$$

(v) Characteristic equation is $z^2 - \frac{1}{4}z - \frac{1}{8} = 0$ with roots

$$z = -\frac{1}{4}, \frac{1}{2} \text{ so that } y_h(n) = A_1\left(-\frac{1}{4}\right)^n + A_2\left(\frac{1}{2}\right)^n$$

Solving for A_1 and A_2 gives

$$y_h(n) = -\frac{1}{4}\left(-\frac{1}{4}\right)^n u(n) + \frac{1}{4}\left(\frac{1}{2}\right)^n u(n)$$

6.19 (i) Ch. Eqn. is $z^2 + \frac{1}{2}z - \frac{1}{6} = 0$, Roots are $\frac{1}{3}, -\frac{1}{2}$

$$y_n(n) = A\left(\frac{1}{3}\right)^n + B\left(-\frac{1}{2}\right)^n$$

$$\text{Let } y_p(n) = k_1 \cos \frac{3n}{4} + k_2 \sin \frac{3n}{4}$$

$$y_p(n-1) = k_1 \cos \frac{3(n-1)}{4} + k_2 \sin \frac{3(n-1)}{4}$$

$$= k_1 \left[\cos \frac{3\pi n}{4} \cos \frac{3\pi}{4} + \sin \frac{3\pi n}{4} \sin \frac{3\pi}{4} \right]$$

$$+ k_2 \left[\sin \frac{3\pi n}{4} \cos \frac{3\pi}{4} - \cos \frac{3\pi n}{4} \sin \frac{3\pi}{4} \right]$$

$$y_p(n-2) = k_1 \left[\cos \frac{3\pi n}{4} \cos \frac{3\pi}{2} + \sin \frac{3\pi n}{4} \sin \frac{3\pi}{2} \right] \\ + k_2 \left[\cos \frac{3\pi}{2} \sin \frac{3\pi n}{4} - \cos \frac{3\pi n}{4} \sin \frac{3\pi}{2} \right]$$

Use $\cos \frac{3\pi}{4} = \frac{1}{\sqrt{2}} = -\sin \frac{3\pi}{4}$, $\cos \frac{3\pi}{2} = 0$, $\sin \frac{3\pi}{2} = -1$

and substitute in d.e. to get

$$k_1 - \frac{k_1}{6\sqrt{2}} - \frac{k_2}{6\sqrt{2}} - \frac{k_2}{2} = 2 - \frac{1}{\sqrt{2}}$$

$$k_2 + \frac{k_1}{6\sqrt{2}} - \frac{k_2}{6\sqrt{2}} + \frac{k_1}{2} = \frac{1}{\sqrt{2}}$$

$$\begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \frac{1}{38 - 5\sqrt{2}} \begin{bmatrix} 78 - 21\sqrt{2} \\ -12 + 9\sqrt{2} \end{bmatrix}$$

$$\therefore y(n) = A_1 \left(-\frac{1}{2}\right)^n + A_2 \left(\frac{1}{3}\right)^n + k_1 \cos \frac{3\pi n}{4} + k_2 \sin \frac{3\pi n}{4}$$

$$\left. \begin{array}{l} n = -1 \text{ gives } -5A - 3B + \frac{15}{16} = 1 \\ n = -2 \text{ gives } 25A + 9B + \frac{45}{16} = 1 \end{array} \right\} \begin{array}{l} A = -\frac{13}{80} \\ B = \frac{1}{4} \end{array}$$

(v) $y_p(n) = k\left(\frac{1}{4}\right)^n$

Substitute in d.e. to get $k = \frac{-51}{2}$

$$y(n) = A \left(-\frac{1}{4}\right)^n + B \left(\frac{1}{3}\right)^n - \frac{51}{2} \left(\frac{1}{4}\right)^n$$

$$y(0) = y(1) = 0 \text{ gives } A = \frac{13}{2}, B = \frac{153}{7}$$

6.20 (i) For $n \geq 1$, $y(n) - y(n-1) + \frac{1}{2} y(n-2) = 0$

so that $h(n) = A_1 e^{-j\frac{\pi}{4}n} + A_2 e^{j\frac{\pi}{4}n}$.

With $y(-1) = 0$, $y(0) = y(-1) - \frac{1}{2} y(-2) + 1 = 1$, solving for the constants gives $A_1 = A_2^* = \frac{1}{\sqrt{2}} e^{j\frac{\pi}{4}}$

so that

$$h(n) = \frac{1}{\sqrt{2}} e^{-j\frac{\pi}{4}(n-1)} + \frac{1}{\sqrt{2}} e^{j\frac{\pi}{4}(n-1)} = \sqrt{2} \cos\left(\frac{\pi}{4}(n-1)\right) u(n)$$

(ii) For $n \geq 2$, $y(n) - \frac{1}{6} y(n-1) - \frac{1}{6} y(n-2) = 0$

so that $h(n) = A_1 \left(-\frac{1}{3}\right)^n + A_2 \left(\frac{1}{2}\right)^n$.

With $y(0) = 1$, we get $y(1) = \frac{1}{6} y(0) - \frac{1}{2} x(0) = -\frac{1}{3}$.

Solving for the constants gives $A_1 = -\frac{1}{5}$, $A_2 = \frac{6}{5}$

so that

$$h(n) = -\frac{1}{5} \left(-\frac{1}{3}\right)^n u(n) + \frac{6}{5} \left(\frac{1}{2}\right)^n u(n)$$

(iii) For $n \geq 1$, $y(n) - y(n-1) + \frac{1}{4} y(n-2) = 0$

so that $h(n) = A_1 \left(\frac{1}{2}\right)^n + A_2 n \left(\frac{1}{2}\right)^n$.

With $y(-1) = 0$, we have $y(0) = x(0) = 1$. Solving for the constants gives $A_1 = A_2 = 1$, so that

$$h(n) = (n+1) \left(\frac{1}{2}\right)^n u(n)$$

(iv) For $n \geq 1$, $y(n) - \frac{3}{2} y(n-1) + \frac{1}{2} y(n-2) = 0$

so that $h(n) = A_1 + A_2 \left(\frac{1}{2}\right)^n$.

$$6.21 \text{ (i)} \quad y(n) = A \left(\frac{1}{3}\right)^n + B \left(-\frac{1}{2}\right)^n \quad \text{for } n \geq 2$$

$$\left. \begin{array}{l} y(0) = 1 \\ y(1) = \frac{1}{3} \end{array} \right\} h(n) = \left(\frac{1}{3}\right)^n u(n) \quad \text{Since } A=1, B=0$$

$$\text{(ii)} \quad y_n(n) = A \left(-\frac{1}{2}\right)^n \quad \text{for } n \geq 1$$

$$y(0) = 1 \quad \therefore h(n) = \left(\frac{1}{2}\right)^n u(n)$$

(iii) Same as (ii)

$$\text{(iv)} \quad y_n(n) = A \left(-\frac{1}{5}\right)^n + B \left(-\frac{1}{3}\right)^n \quad \text{for } n \geq 1$$

$$\left. \begin{array}{l} y(0) = 1 \\ y(-1) = 0 \end{array} \right\} \therefore \begin{array}{l} A = \frac{3}{2} \\ B = \frac{5}{2} \end{array} \quad \therefore h(n) = \frac{3}{2} \left(-\frac{1}{5}\right)^n + \frac{5}{2} \left(-\frac{1}{3}\right)^n$$

for $n \geq 0$

(v) Same as 6.18 (v)

6.22 Ch. eqn. of the assumed eqn is $z^2 + az + b = 0$

From given $h(n)$, roots of ch. eqn: must be $\frac{1}{2}, -\frac{1}{2}$

$$\therefore (z - \frac{1}{2})(z + \frac{1}{2}) = z^2 + az + b, \quad a = 0, \quad b = -\frac{1}{4}$$

For $n \geq 2$, eqn is $y(n) - \frac{1}{4} y(n-2) = 0$

$$\text{with solution } y(n) = A(\frac{1}{2})^n + B(-\frac{1}{2})^n$$

$$\text{I.C. are } y(0) = C, \quad y(1) = d$$

$$\text{Since } A = B = 1, \quad C = 2, \quad d = 0$$

$$\therefore \text{Eqn is } y(n) - \frac{1}{4} y(n-2) = 2x(n)$$

6.23 Let $h_1(n) = (-\frac{1}{2})^n u(n)$

$$\text{Then } h(n) = h_1(n) + \frac{1}{2} h_1(n-1)$$

clearly $h_1(n)$ is impulse response of a system of the form

$$y(n) + \frac{1}{2} y(n-1) = x(n)$$

$$\therefore h(n) \text{ corresponds to } y(n) + \frac{1}{2} y(n-1) = x(n) + \frac{1}{2} x(n-1)$$

6.24(a) For $n \geq 1$, we have $\sum_{k=0}^N a_k y(n-k) = 0$

$$y(0) = \frac{1}{a_0} [x(0) - \sum_{k=1}^N a_k y(-k)] = \frac{1}{a_0}$$

(b) Let $y(n) - \frac{3}{4} y(n-1) + \frac{1}{8} y(n-2) = x(n)$ have impulse response $h^*(n)$

$$\text{Then } h^*(n) = A(\frac{1}{2})^n - B(\frac{1}{4})^n$$

$$\text{with } \begin{cases} h^*(0) = 1 \\ h^*(-1) = 0 \end{cases} \quad \text{Gives } A = 2, \quad B = 1$$

$$\therefore h^*(n) = [2(\frac{1}{2})^n - (\frac{1}{4})^n] u(n)$$

$$\therefore h(n) = [2(\frac{1}{2})^n - (\frac{1}{4})^n] u(n) + \frac{1}{2} [2(\frac{1}{2})^{n-1} - (\frac{1}{4})^{n-1}] u(n-1)$$

$$= \begin{cases} 1 & n=0 \\ 4\left(\frac{1}{2}\right)^n - 3\left(\frac{1}{4}\right)^n & n \geq 1 \end{cases} = \left[4\left(\frac{1}{2}\right)^n - 3\left(\frac{1}{4}\right)^n \right] u(n)$$

(c)(i) Need to solve $y(n) + y(n-1) + y(n-2) = 0$
with $y(-1) = 0$, $y(0) = 1$

Gives $y(n) = A \cos \frac{2\pi}{3}n + B \sin \frac{2\pi}{3}n$

$$\begin{cases} y(-1) = 0 = \frac{1}{2}A - \frac{\sqrt{3}}{2}B \\ y(0) = 1 = A \end{cases} \quad \left. \begin{array}{l} A = 1 \\ B = \frac{1}{\sqrt{3}} \end{array} \right\}$$

$$\therefore h(n) = \left[\cos \frac{2\pi}{3}n + \frac{1}{\sqrt{3}} \sin \frac{2\pi}{3}n \right] u(n)$$

(ii) To find $h(n)$, solve $y(n) - \frac{1}{4}y(n-1) - \frac{1}{8}y(n-2) = 0$
for $n \geq 1$ with $y(-1) = 0$, $y(0) = 1$

$$\therefore h(n) = \left[\frac{2}{3}\left(\frac{1}{2}\right)^n - \frac{1}{3}\left(-\frac{1}{4}\right)^n \right] u(n)$$

$$\begin{aligned} \therefore h(n) &= \left[\frac{2}{3}\left(\frac{1}{2}\right)^n + \frac{1}{3}\left(-\frac{1}{4}\right)^n \right] u(n) - \frac{1}{2} \left[\frac{2}{3}\left(\frac{1}{2}\right)^{n-1} + \frac{1}{3}\left(-\frac{1}{4}\right)^{n-1} \right] u(n-1) \\ &= \left(-\frac{1}{4}\right)^n u(n) \end{aligned}$$

(iii) Solve $y(n) - y(n-1) + \frac{15}{64}y(n-2) = 0$ $n \geq 1$
 $y(-1) = 0$, $y(0) = 1$

gives $h(n) = \left[\frac{5}{2}\left(\frac{5}{8}\right)^n - \frac{3}{2}\left(\frac{3}{8}\right)^n \right] u(n)$

(iv) Solve $y(n+2) + \frac{2}{3}y(n+1) + \frac{1}{9}y(n) = 0$ for $n \geq 1$
with $y(1) = 0$, $y(2) = 1$

Solution is $y(n) = A\left(-\frac{1}{3}\right)^n + Bn\left(-\frac{1}{3}\right)^n$

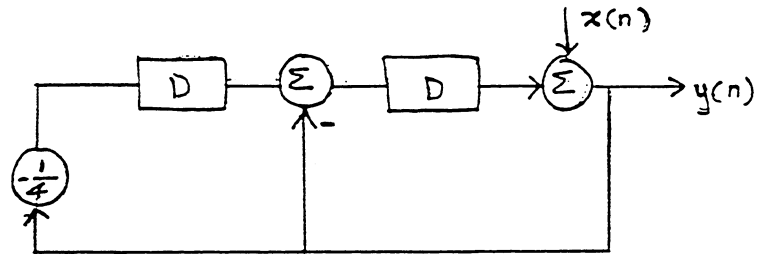
$$\begin{cases} n=1 \text{ gives } -\frac{1}{3}A - \frac{1}{3}B = 0 \\ n=2 \text{ gives } -\frac{1}{9}A + \frac{2}{9}B = 1 \end{cases} \quad \left. \begin{array}{l} A = -9 \\ B = 9 \end{array} \right\}$$

$$\therefore h(n) = \left[-9\left(-\frac{1}{3}\right)^n + 9n\left(-\frac{1}{3}\right)^n \right] u(n)$$

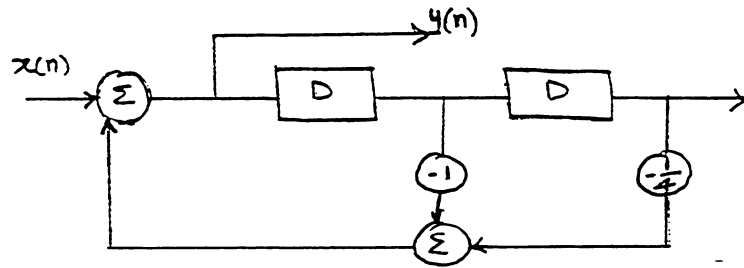
(v) For $y(n] = x(n)$, we have $h^*(n) = \delta(n)$

$$\begin{aligned} \therefore h(n) &= h^*(n) - 3h^*(n-1) + 2h^*(n-2) - h^*(n-3) \\ &= \delta(n) - 3\delta(n-1) + 2\delta(n-2) - \delta(n-3) \end{aligned}$$

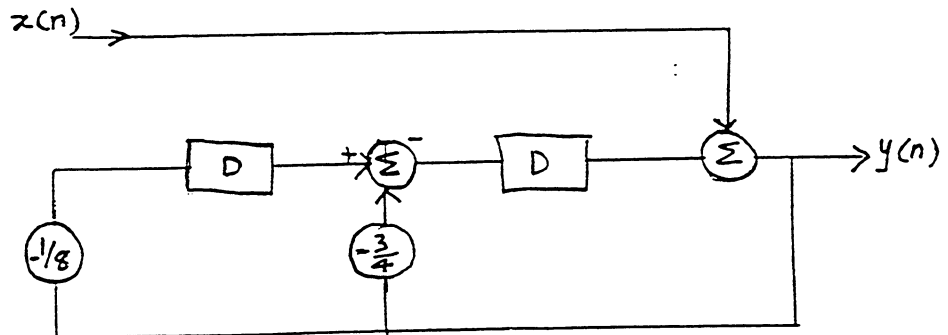
6.25 (i) (a)



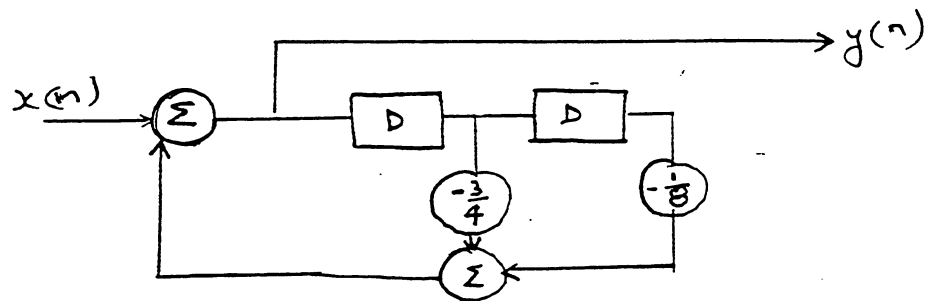
(b)

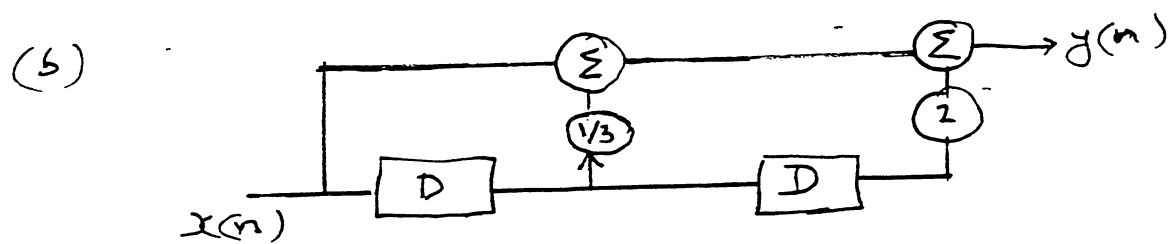
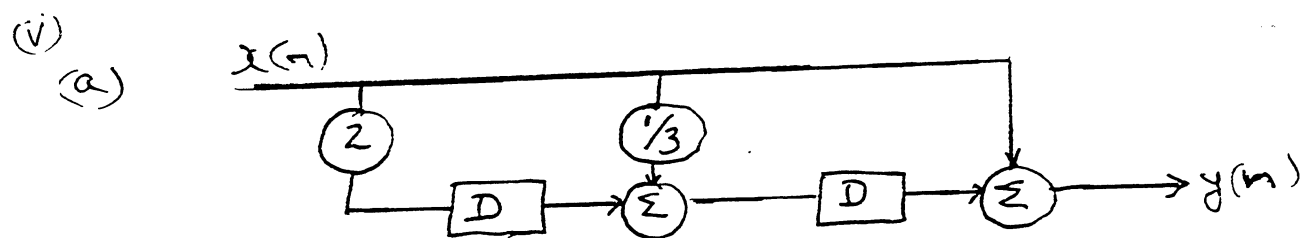
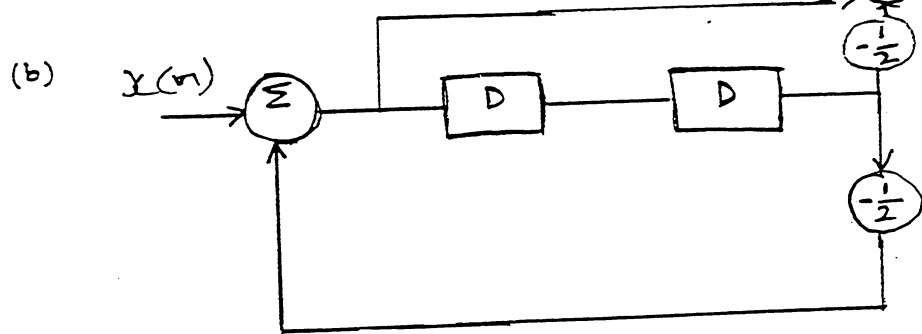
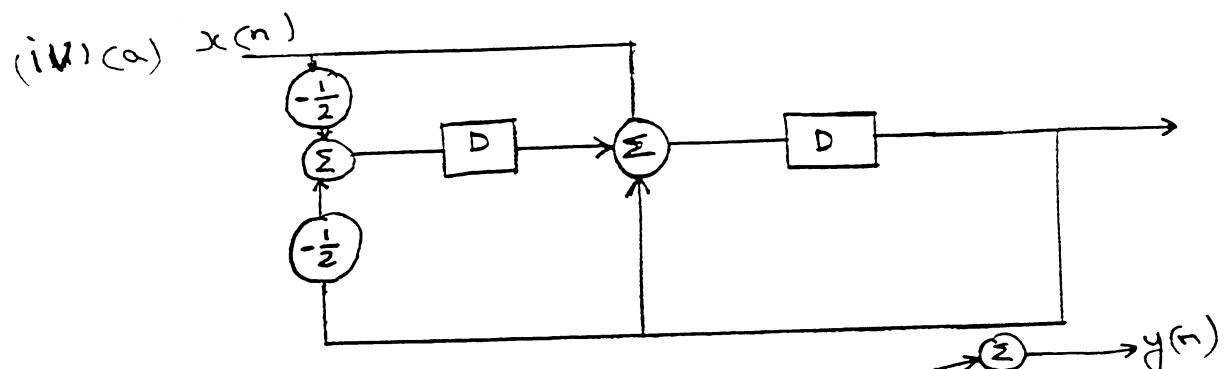


(ii)
(iii) (a)



(b)





6.26

(i) (a)

$$y(n) = v_1(n) + x(n)$$

$$v_1(n+1) = v_2(n) - y(n) = -v_1(n) + v_2(n) - x(n)$$

$$v_2(n+1) = -\frac{1}{4} y(n) = -\frac{1}{4} v_1(n) - \frac{1}{4} x(n)$$

(b)

$$\hat{v}_1(n+1) = \hat{v}_2(n)$$

$$\hat{v}_2(n+1) = -\frac{1}{4} \hat{v}_1(n) - \hat{v}_2(n) + x(n)$$

$$y(n) = -\frac{1}{4} \hat{v}_1(n) - \hat{v}_2(n) + x(n)$$

(ii) (a)

$$y(n) = v_1(n) + x(n)$$

(iii)

$$v_1(n+1) = v_2(n) - \frac{3}{4} y(n) = -\frac{3}{4} v_1(n) + v_2(n) - \frac{3}{4} x(n)$$

$$v_2(n+1) = -\frac{1}{8} y(n) = -\frac{1}{8} v_1(n) - \frac{1}{8} x(n)$$

(b)

$$\hat{v}_1(n+1) = \hat{v}_2(n)$$

$$\hat{v}_2(n+1) = -\frac{1}{8} \hat{v}_1(n) - \frac{3}{4} \hat{v}_2(n) + x(n)$$

$$y(n) = -\frac{1}{8} \hat{v}_1(n) - \frac{3}{4} \hat{v}_2(n) + x(n)$$

(iv) (a)

$$y(n) = v_1(n)$$

$$v_1(n+1) = v_2(n) + x(n)$$

$$v_2(n+1) = -\frac{1}{2} v_1(n) - \frac{1}{2} x(n)$$

(b)

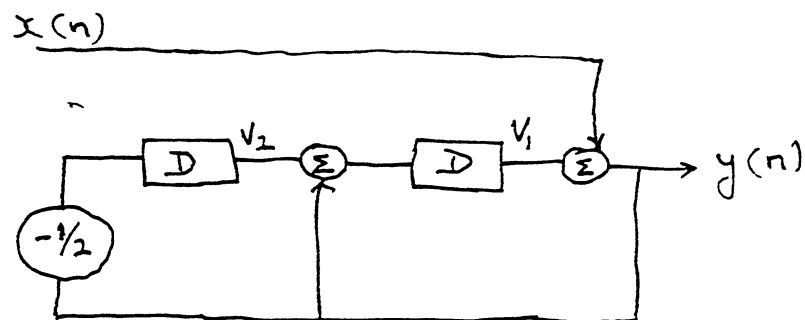
$$\hat{v}_1(n+1) = \hat{v}_2(n)$$

$$\hat{v}_2(n+1) = -\frac{1}{2} \hat{v}_1(n) + x(n)$$

$$y(n) = -\frac{1}{2} \hat{v}_1(n) + \hat{v}_2(n)$$

(v) System has no states.

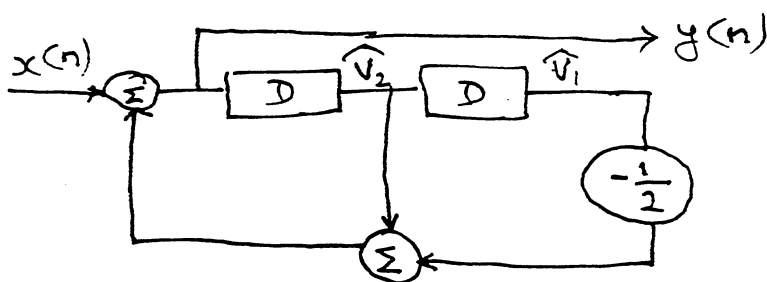
6.27 (i)



$$y(n) = v_1(n) + x(n)$$

$$v_1(n+1) = v_2(n) + y(n) = v_1(n) + v_2(n) + x(n)$$

$$v_2(n+1) = -\frac{1}{2} y(n) = -\frac{1}{2} v_1(n) - \frac{1}{2} x(n)$$

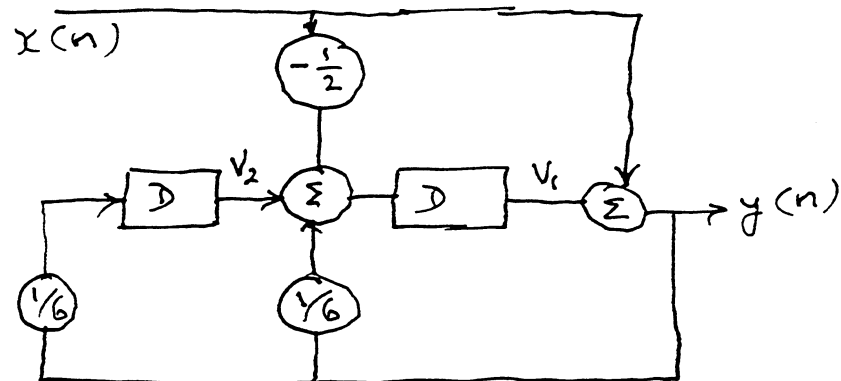


$$\hat{v}_1(n+1) = \hat{v}_2(n)$$

$$\hat{v}_2(n+1) = x(n) - \frac{1}{2} \hat{v}_1(n) + \hat{v}_2(n)$$

$$y(n) = -\frac{1}{2} \hat{v}_1(n) + \hat{v}_1(n) + x(n)$$

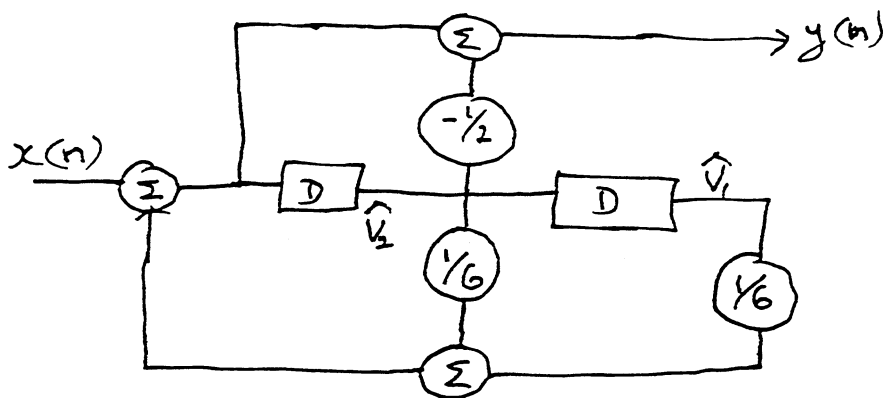
(ii)



$$y(n) = v_1(n) + x(n)$$

$$v_1(n+1) = v_2(n) + \frac{1}{6}y(n) - \frac{1}{2}x(n) = \frac{1}{6}v_1(n) + v_2(n) - \frac{1}{3}x(n)$$

$$v_2(n+1) = \frac{1}{6}y(n) = \frac{1}{6}v_1(n) + \frac{1}{6}x(n)$$



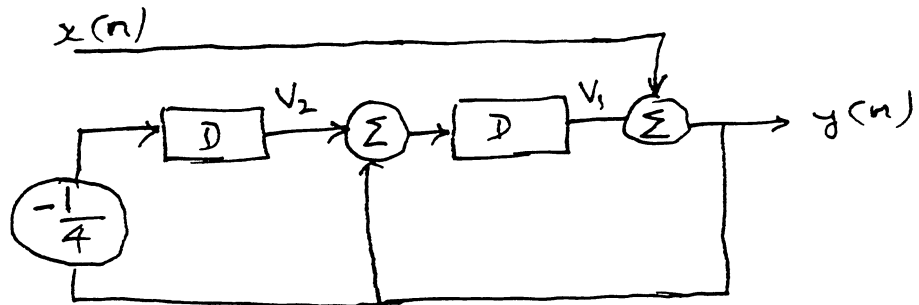
$$\hat{v}_1(n+1) = \hat{v}_2(n)$$

$$\hat{v}_2(n+1) = \frac{1}{6}\hat{v}_1(n) + \frac{1}{6}\hat{v}_2(n) + x(n)$$

$$y(n) = -\frac{1}{2}\hat{v}_2(n) + \hat{v}_2(n+1)$$

$$= \frac{1}{6}\hat{v}_1(n) - \frac{1}{3}\hat{v}_2(n) + x(n)$$

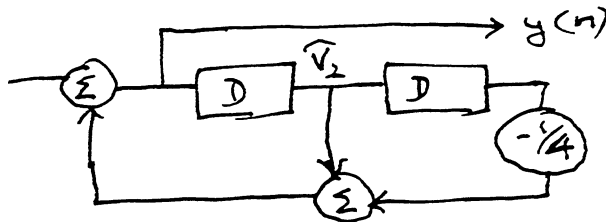
(iii)



$$y(n) = v_1(n) + x(n)$$

$$v_1(n+1) = v_2(n) + y(n) = v_1(n) + v_2(n) + x(n)$$

$$v_2(n+1) = -\frac{1}{4} y(n) = -\frac{1}{4} v_1(n) - \frac{1}{4} x(n)$$

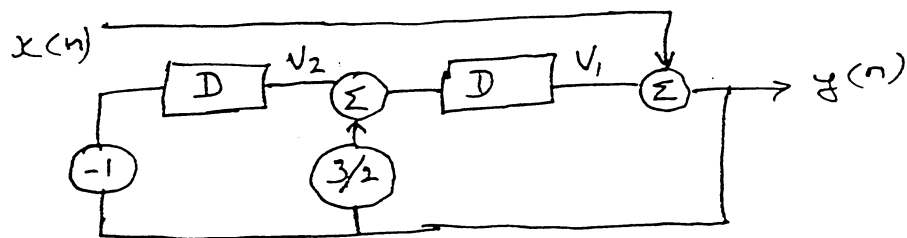


$$\hat{v}_1(n+1) = \hat{v}_2(n)$$

$$\hat{v}_2(n+1) = -\frac{1}{4} \hat{v}_1(n) + \hat{v}_2(n) + x(n)$$

$$y(n) = -\frac{1}{4} \hat{v}_1(n) + \hat{v}_2(n) + x(n)$$

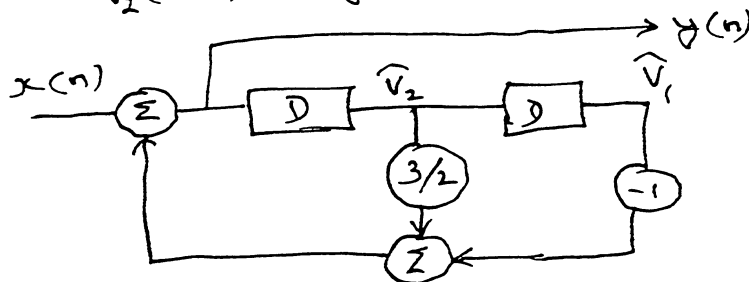
(iv)



$$y(n) = v_1(n) + x(n)$$

$$v_1(n+1) = v_2(n) + \frac{3}{2}y(n) = \frac{3}{2}v_1(n) + v_2(n) + x(n)$$

$$v_2(n+1) = -y(n) = -v_1(n) - x(n)$$

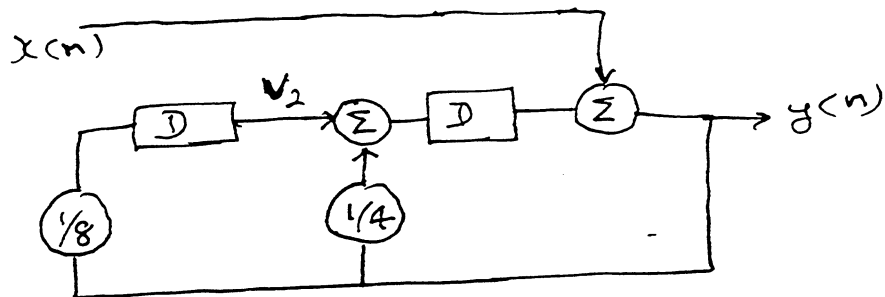


$$\hat{v}_1(n+1) = \hat{v}_2(n)$$

$$\hat{v}_2(n+1) = -\hat{v}_1(n) + \frac{3}{2}\hat{v}_2(n) + x(n)$$

$$y(n) = -\hat{v}_1(n) + x(n)$$

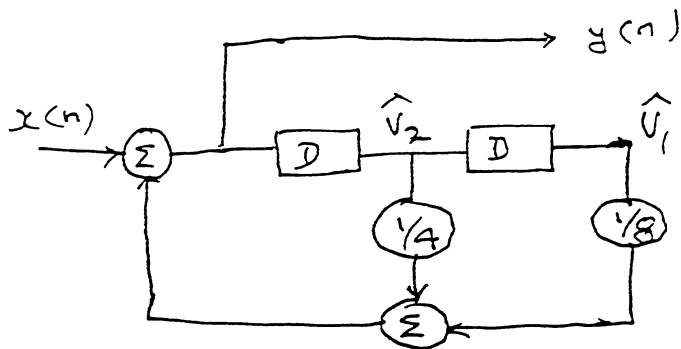
(v)



$$y(n) = v_1(n) + x(n)$$

$$v_1(n+1) = v_2(n) + \frac{1}{4}y(n) = \frac{1}{4}v_1(n) + v_2(n) + x(n)$$

$$v_2(n+1) = \frac{1}{8}y(n) = \frac{1}{8}v_1(n) + \frac{1}{8}x(n)$$



$$\hat{v}_1(n+1) = \hat{v}_2(n)$$

$$\hat{v}_2(n+1) = \frac{1}{8}\hat{v}_1(n) + \frac{1}{4}\hat{v}_2(n) + x(n)$$

$$y(n) = \frac{1}{8}\hat{v}_1(n) + \frac{1}{4}\hat{v}_2(n) + x(n)$$

6.28 (i) Use Second Canonical form to get

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{1}{24} & -\frac{9}{24} & \frac{13}{12} \end{bmatrix} \quad b = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad c = [1 \ 0 \ 0] \quad d = 0$$

Characteristic equation is $z^3 - \frac{13}{12}z^2 + \frac{9}{24}z - \frac{1}{24} = 0$

With roots $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}$

$$\therefore x_0 + \frac{1}{2}x_1 + \frac{1}{4}x_2 = \left(\frac{1}{2}\right)^n$$

$$x_0 + \frac{1}{3}x_1 + \frac{1}{9}x_2 = \left(\frac{1}{3}\right)^n$$

$$\alpha_0 + \frac{1}{4}\alpha_1 + \frac{1}{16}\alpha_2 = \left(\frac{1}{4}\right)^n$$

$$\text{So that } \alpha_0 = 2\left(\frac{1}{2}\right)^n - 9\left(\frac{1}{3}\right)^n + 8\left(\frac{1}{4}\right)^n$$

$$\alpha_1 = -14\left(\frac{1}{2}\right)^n + 54\left(\frac{1}{3}\right)^n - 40\left(\frac{1}{4}\right)^n$$

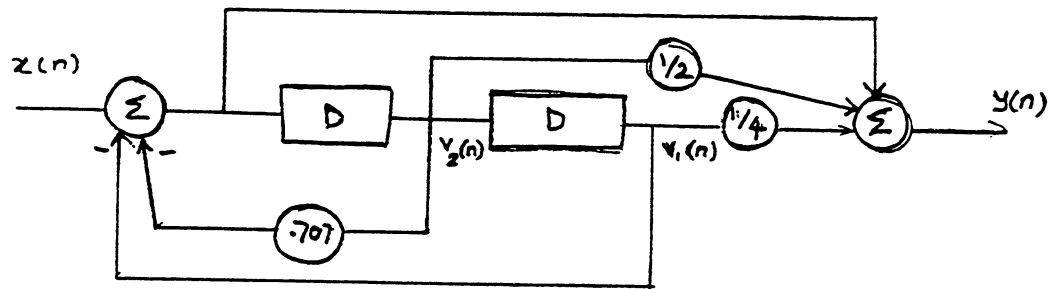
$$\alpha_2 = 24\left(\frac{1}{2}\right)^n - 72\left(\frac{1}{3}\right)^n + 48\left(\frac{1}{4}\right)^n$$

$$A^n = \alpha_0 I + \alpha_1 A + \alpha_2 A^2$$

$$= \alpha_0 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \alpha_1 \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{1}{24} & -\frac{9}{24} & \frac{13}{12} \end{bmatrix} + \alpha_2 \begin{bmatrix} 0 & 0 & 1 \\ \frac{1}{24} & -\frac{9}{24} & \frac{13}{12} \\ \frac{13}{288} & -\frac{35}{96} & \frac{115}{144} \end{bmatrix}$$

$$= \begin{bmatrix} 2\left(\frac{1}{2}\right)^n - 9\left(\frac{1}{3}\right)^n + 8\left(\frac{1}{4}\right)^n & -14\left(\frac{1}{2}\right)^n + 54\left(\frac{1}{3}\right)^n - 40\left(\frac{1}{4}\right)^n & 24\left(\frac{1}{2}\right)^n - 72\left(\frac{1}{3}\right)^n + 48\left(\frac{1}{4}\right)^n \\ \left(\frac{1}{2}\right)^n - 3\left(\frac{1}{3}\right)^n - 4\left(\frac{1}{4}\right)^n & -7\left(\frac{1}{2}\right)^n + 18\left(\frac{1}{3}\right)^n - 10\left(\frac{1}{4}\right)^n & 12\left(\frac{1}{2}\right)^n - 24\left(\frac{1}{3}\right)^n + 12\left(\frac{1}{4}\right)^n \\ \frac{1}{2}\left(\frac{1}{2}\right)^n - \left(\frac{1}{3}\right)^n + \frac{1}{2}\left(\frac{1}{4}\right)^n & -\frac{7}{2}\left(\frac{1}{2}\right)^n + 6\left(\frac{1}{3}\right)^n - \frac{5}{2}\left(\frac{1}{4}\right)^n & 6\left(\frac{1}{2}\right)^n - 8\left(\frac{1}{3}\right)^n + 3\left(\frac{1}{4}\right)^n \end{bmatrix}$$

(ii) Second Canonical form gives



$$\therefore v_1(n+1) = v_2(n)$$

$$v_2(n+1) = -v_1(n) - 0.707 v_2(n)$$

$$y(n) = -0.75 v_1(n) - 0.207 v_2(n) + x(n)$$

$$\therefore A = \begin{bmatrix} 0 & 1 \\ -0.75 & -0.707 \end{bmatrix} \quad \underline{b} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad \underline{c} = [-0.75 \quad -0.207] \quad d = 1$$

Characteristic equation is $z^2 + \frac{1}{\sqrt{2}}z + 1 = 0$

with roots $z = e^{\pm j\theta}$ where $\theta = \tan^{-1}(-\sqrt{7})$

$$\therefore \alpha_0 + \alpha_1 e^{j\theta} = e^{jn\theta}$$

$$\alpha_0 + \alpha e^{-j\theta} = e^{-jn\theta}$$

$$\text{So that } \alpha_0 = \frac{-\sin(n-1)\theta}{\sin\theta} \quad \text{where } \sin\theta = \frac{1}{2\sqrt{2}}$$

$$\alpha_1 = \frac{\sin n\theta}{\sin\theta}$$

$$\therefore A^n = -\frac{\sin(n-1)\theta}{\sin\theta} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \frac{\sin n\theta}{\sin\theta} \begin{bmatrix} 0 & 1 \\ -\frac{3}{4} & -\frac{1}{\sqrt{2}} \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{\sin(n-1)\theta}{\sin\theta} & \frac{\sin n\theta}{\sin\theta} \\ -\frac{3}{4} \frac{\sin n\theta}{\sin\theta} & -\frac{\sin(n-1)\theta}{\sin\theta} - \frac{1}{\sqrt{2}} \frac{\sin n\theta}{\sin\theta} \end{bmatrix}$$

(iii) Second Canonical form gives

$$A = \begin{bmatrix} 0 & 1 \\ -2 & 3 \end{bmatrix} \quad \underline{b} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad \underline{c} = [1 \ 0] \quad d = 0$$

Characteristic eqn is $z^2 - 3z + 2 = 0$

with roots $z = +1, +2$

$$\therefore \alpha_0 + \alpha_1 = 1 \quad \text{so that } \alpha_0 = 2 - 2^n$$

$$\alpha_0 + 2\alpha_1 = 2^n \quad \alpha_1 = 2^n - 1$$

$$A^n = (2 - 2^n) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + (2^n - 1) \begin{bmatrix} 0 & 1 \\ -2 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 2 - 2^n & 2^n - 1 \\ -2^{n+1} + 2 & -1 + 2^{n+1} \end{bmatrix}$$

$$6.29 \quad \underline{v}(n) = \sum_{j=0}^{n-1} A^{n-j-1} b$$

$$(i) \quad \underline{v}(n) = \sum_{k=0}^{n-1} \begin{bmatrix} 24\left(\frac{1}{2}\right)^k - 72\left(\frac{1}{3}\right)^k + 48\left(\frac{1}{4}\right)^k \\ 12\left(\frac{1}{2}\right)^k - 24\left(\frac{1}{3}\right)^k + 12\left(\frac{1}{4}\right)^k \\ 6\left(\frac{1}{2}\right)^k - 8\left(\frac{1}{3}\right)^k + 3\left(\frac{1}{4}\right)^k \end{bmatrix}$$

$$= \begin{bmatrix} 48 \left[1 - \left(\frac{1}{2}\right)^n\right] - 108 \left[1 - \left(\frac{1}{3}\right)^n\right] + 64 \left[1 - \left(\frac{1}{4}\right)^n\right] \\ 24 \left[1 - \left(\frac{1}{2}\right)^n\right] - 36 \left[1 - \left(\frac{1}{3}\right)^n\right] + 16 \left[1 - \left(\frac{1}{4}\right)^n\right] \\ 12 \left[1 - \left(\frac{1}{2}\right)^n\right] - 12 \left[1 - \left(\frac{1}{3}\right)^n\right] + 4 \left[1 - \left(\frac{1}{4}\right)^n\right] \end{bmatrix}$$

$$y(n) = v_1(n) = 4 - 48\left(\frac{1}{2}\right)^n + 108\left(\frac{1}{3}\right)^n - 64\left(\frac{1}{4}\right)^n$$

$$(ii) \quad \underline{v}(n) = \sum_{k=0}^{n-1} \begin{bmatrix} \frac{\sin k\theta}{\sin \theta} \\ -\frac{\sin(k-1)\theta}{\sin \theta} - \frac{1}{\sqrt{2}} \frac{\sin k\theta}{\sin \theta} \end{bmatrix} \quad n \geq 0$$

$$= \frac{1}{2j \sin \theta} \sum_{k=0}^{n-1} \begin{bmatrix} e^{jk\theta} & -e^{-jk\theta} \\ -e^{j(k-1)\theta} + e^{-j(k-1)\theta} & -\frac{1}{\sqrt{2}} e^{jk\theta} + \frac{1}{\sqrt{2}} e^{-jk\theta} \end{bmatrix}$$

$$= \frac{1}{2j \sin \theta} \begin{bmatrix} \frac{1-e^{jn\theta}}{1-e^{j\theta}} - \frac{1-e^{-jn\theta}}{1-e^{-j\theta}} \\ -e^{-j\theta} \frac{1-e^{jn\theta}}{1-e^{j\theta}} + e^{j\theta} \frac{1-e^{-jn\theta}}{1-e^{-j\theta}} - \frac{1}{\sqrt{2}} \frac{1-e^{jn\theta}}{1-e^{j\theta}} + \frac{1}{\sqrt{2}} \frac{1-e^{-jn\theta}}{1-e^{-j\theta}} \end{bmatrix}$$

$$= \frac{1}{2 \sin \theta [-\cos \theta]} \begin{bmatrix} \sin \theta - \sin n\theta + \sin (n-1)\theta \\ \left(1 - \frac{1}{\sqrt{2}}\right) \sin \theta - \sin 2\theta + \left(1 - \frac{1}{\sqrt{2}}\right) \sin (n-1)\theta + \frac{1}{\sqrt{2}} \sin n\theta \\ - \sin (n-2)\theta \end{bmatrix}$$

$$y(n) = -\frac{3}{4} v_1(n) + \left(\frac{1}{2} - \frac{1}{\sqrt{2}}\right) v_2(n) + x(n) \quad n \geq 0$$

$$\begin{aligned} \text{(iii)} \quad y(n) &= \sum_{k=0}^{n-1} \begin{bmatrix} 2^k - 1 \\ -1 + 2^{k+1} \end{bmatrix} \\ &= \begin{bmatrix} 2^n - 1 - n \\ 2^{n+1} - 2 - n \end{bmatrix} \end{aligned}$$

$$y(n) = 2^n - 1 - n \quad n \geq 0$$

$$6.30 \quad (a) \quad |\lambda I - A| = \begin{vmatrix} \lambda - 1 & -\frac{1}{2} \\ \frac{5}{3} & \lambda + \frac{7}{6} \end{vmatrix} = \lambda^2 + \frac{1}{6}\lambda - \frac{1}{3} = \left(\lambda - \frac{1}{2}\right)\left(\lambda + \frac{2}{3}\right)$$

Characteristic roots are $\lambda = \frac{1}{2}, -\frac{2}{3}$

(b) Let $P = \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix}$. Then $PA = \hat{A}P$ gives

$$\begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix} \begin{bmatrix} 1 & \frac{1}{2} \\ -\frac{5}{3} & -\frac{7}{6} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{2}{3} \end{bmatrix} \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix}$$

$$\left. \begin{aligned} p_{11} - \frac{5}{3} p_{12} &= \frac{1}{2} p_{11} \\ \frac{1}{2} p_{11} - \frac{7}{6} p_{12} &= \frac{1}{2} p_{12} \\ p_{21} - \frac{5}{3} p_{22} &= -\frac{2}{3} p_{21} \\ \frac{1}{2} p_{21} - \frac{7}{6} p_{22} &= -\frac{1}{3} p_{22} \end{aligned} \right\} \begin{aligned} &\text{Gives } p_{11} = \frac{10}{3} p_{12} \\ &p_{21} = p_{22} \\ &\text{Let } p_{11} = p_{22} = 1 \end{aligned}$$

$$\therefore P = \begin{bmatrix} 1 & 3/10 \\ 1 & 1 \end{bmatrix}, \quad P^{-1} = \begin{bmatrix} 10/7 & -3/7 \\ -10/7 & 10/7 \end{bmatrix}$$

$$\hat{\underline{b}} = P^{-1} \underline{b} = \begin{bmatrix} 10/7 & -3/7 \\ -10/7 & 10/7 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 13/7 \\ -20/7 \end{bmatrix}$$

$$\hat{\underline{c}} = \underline{c} P = [1 \ 1] \begin{bmatrix} 1 & 3/10 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 13/10 \end{bmatrix}$$

$$\hat{\underline{d}} = \underline{d} = 0$$

$$\hat{\underline{V}}(0) = P^{-1} \underline{V}(0) = \begin{bmatrix} \frac{10}{7} V_1(0) - \frac{3}{7} V_2(0) \\ -\frac{10}{7} V_1(0) + \frac{10}{7} V_2(0) \end{bmatrix}$$

$$\Leftrightarrow \begin{bmatrix} \hat{V}_1(n+1) \\ \hat{V}_2(n+1) \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{2}{3} \end{bmatrix} \begin{bmatrix} \hat{V}_1(n) \\ \hat{V}_2(n) \end{bmatrix} + \begin{bmatrix} 13/7 \\ -20/7 \end{bmatrix} x(n)$$

$$\hat{\underline{A}}^n = \begin{bmatrix} (\frac{1}{2})^n & 0 \\ 0 & (-\frac{2}{3})^n \end{bmatrix}. \quad \text{Assume } \hat{\underline{V}}(0) = 0$$

$$\therefore \begin{bmatrix} \hat{V}_1(n) \\ \hat{V}_2(n) \end{bmatrix} = \sum_{j=0}^{n-1} \begin{bmatrix} (\frac{1}{2})^n & 0 \\ 0 & (-\frac{2}{3})^n \end{bmatrix} \begin{bmatrix} 13/7 \\ -20/7 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{26}{7} [1 - (\frac{1}{2})^n] \\ -\frac{12}{7} [1 - (-\frac{2}{3})^n] \end{bmatrix} \quad n \geq 0$$

$$(d) \underline{\hat{V}}(n) = P \hat{\underline{V}}(n) = \begin{bmatrix} \frac{16}{5} - \frac{26}{7} (\frac{1}{2})^n + \frac{18}{35} (-\frac{2}{3})^n \\ 2 - \frac{26}{7} (\frac{1}{2})^n + \frac{12}{7} (-\frac{2}{3})^n \end{bmatrix}$$

6.31 Consider the system described by

$$y_n + \sum_{j=1}^N a_j y(n-j) = \sum_{m=0}^N b_m x(n-m)$$

The characteristic equation for this system is

$$1 + \sum_{j=1}^N a_j z^{-j} = 0 \quad \text{or equivalently}$$

$$z^N + a_1 z^{N-1} + \dots + a_N = 0$$

The A matrix in the second form is given by

$$\begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & & & & \\ 0 & & & & 1 \\ -a_N & & & & -a_1 \end{bmatrix}$$

so that we must show that $\det |zI - A| = z^N + \sum_{j=1}^N a_j z^{N-j}$

We can prove this by induction. Clearly the result holds for $N=1$ and 2 since $A = [a_1]$ and $A = \begin{bmatrix} 0 & 1 \\ a_2 & a_1 \end{bmatrix}$

Assume it holds for order $N-1$. Then for N , we have

$$\begin{aligned} \det [zI - A] &= \begin{vmatrix} z & -1 & 0 & \dots & 0 \\ 0 & z & -1 & \dots & 0 \\ \vdots & & & & \\ 0 & & & & z & -1 \\ a_N & a_{N-1} & \dots & a_2 & z+a_1 \end{vmatrix} \\ &= z \begin{vmatrix} z & -1 & \dots & 0 \\ 0 & z & -1 & \dots & 0 \\ \vdots & & & & \\ a_{N-1} & \dots & & z+a_1 \end{vmatrix} + a_N \end{aligned}$$

$$= z \left[z^{N-1} + \sum_{j=1}^{N-1} a_j z^{N-1-j} \right] + a_N$$

$$= z^N + \sum_{j=1}^N a_j z^{N-j}$$

6.32 (i) $\sum_{n=-\infty}^{\infty} |h(n)| = \sum_{n=-\infty}^{-1} \left(\frac{1}{3}\right)^n + \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n$ is not bounded

$\therefore$ unstable

(ii) $\sum_{n=-\infty}^{\infty} |h(n)| = \sum_{n=0}^{100} 3^n = \frac{1}{2} (3^{101} - 1) < \infty$

$\therefore$ stable

(iii) $\sum |h(n)| < \infty$ stable

(iv) $|y(n)| = |x(n) + 2x(n-1) + \frac{1}{2}x(n-2)|$
 $\leq |x(n)| + 2|x(n-1)| + \frac{1}{2}|x(n-2)| < \infty$
 if $|x(n)| < \infty$

stable

(v) Ch. Eqn is $z^2 - 2z + 1 = 0$

Roots are $z = 1, 1$ unstable

(vi) $z^2 - \frac{2}{3}z - \frac{1}{3} = 0$, Roots are $z = -1, \frac{1}{3}$ Unstable

(vii) $\text{Det}(zI - A) = \begin{vmatrix} z - \frac{1}{2} & -\frac{1}{4} \\ \frac{1}{4} & z - \frac{1}{2} \end{vmatrix} = z^2 - z + \frac{5}{16}$

Roots are $z = -\frac{1}{2} \pm \frac{1}{8}\sqrt{11}$ Stable

(viii) $\text{Det}(zI - A) = \begin{vmatrix} z & -1 \\ 2 & z+3 \end{vmatrix} = z^2 + 3z + 2$

Roots are $-2, -1$ Unstable

CHAPTER 7

$$7.1(a) \quad \Omega_0 = \frac{3\pi}{4} \quad N = \frac{2\pi}{3\pi/4} = 8$$

$$x(n) = \frac{1}{2} e^{j\Omega_0 n} + \frac{1}{2} e^{-j\Omega_0 n}$$

$$\therefore a_1 = -a_{-1} = a_2 = \frac{1}{2}j \text{ All other } a_k = 0 \quad 0 \leq k \leq 7$$

$$b) \quad x(n) = 2 \left[\sin \frac{7\pi n}{6} - \sin \frac{\pi n}{6} \right]$$

$$N = 12, \quad \Omega_0 = \pi/6$$

$$\therefore x(n) = \frac{1}{j} \left[e^{j7\Omega_0 n} - e^{-j7\Omega_0 n} - e^{j\Omega_0 n} + e^{-j\Omega_0 n} \right]$$

$$\therefore a_1 = \frac{-1}{j} = j = a_{-1}^* = a_{11}^*$$

$$a_7 = -j = a_{-7}^* = a_5^*$$

All others zero

$$c) \quad e^{j\frac{2\pi}{5}} = 0.3090 + j \cdot 0.9511$$

$$a_k = \frac{1}{5} \left[1 - e^{j\frac{2\pi}{5}k} + e^{j\frac{6\pi}{5}k} - e^{j\frac{8\pi}{5}k} \right]$$

$$a_0 = 0 \quad a_1 = -0.4271 - j0.5878 = a_4^*$$

$$a_2 = 0.9271 + j0.9511 = a_3^*$$

$$d) \quad e^{j\frac{2\pi}{8}} = \frac{1}{\sqrt{2}} + j \frac{1}{\sqrt{2}} = e^{j\frac{\pi}{4}}$$

$$a_k = \frac{1}{8} \left[1 + e^{j\frac{\pi}{4}k} + e^{j\frac{\pi}{2}k} + e^{j\frac{3\pi}{4}k} \right]$$

$$a_0 = \frac{1}{2}, \quad a_1 = 0.125 + j0.0518 = a_7^*$$

$$a_2 = 0 \quad a_3 = 0.125 + j0.0518 = a_5^*, \quad a_4 = 0$$

$$e) \quad e^{j \frac{2\pi}{6}} = \frac{1}{2} + j \frac{\sqrt{3}}{2} = e^{j \pi/3}$$

$$a_k = \left[e^{j \pi/3 k} + 2e^{j \frac{2\pi}{3} k} + 3e^{j \pi k} \right] \frac{1}{8}$$

$$a_0 = 0.75 \quad a_1 = -3.5 + j 2.5981$$

$$= a_5^* = -4.375 + j 3.248$$

$$a_2 = -1.875 - j 1.083 = a_4^*, \quad a_3 = -2.5$$

b) First term is periodic with period 2,
Second term is periodic with period 3.

∴ Period of the sum is $N=6$, $\Omega_0 = \frac{2\pi}{6} = \frac{\pi}{3}$

$$\text{Now } \cos \frac{2\pi n}{3} = \frac{1}{2} \left[e^{j 2\Omega_0 n} + e^{-j 2\Omega_0 n} \right]$$

$$\therefore a_2 = \frac{1}{2} = a_4$$

Also the coefficients of $\sum_{k=-\infty}^{\infty} (-1)^k \delta(n-k)$.

Considered as a periodic sequence with $N=6$

$$\text{is } a_k = \frac{1}{6} \left[1 - e^{j \pi/3 k} + e^{j 2\pi/3 k} - e^{j \pi k} + e^{j 4\pi/3 k} - e^{j 5\pi/3 k} \right]$$

so that $a_3 = 1$, all others zero

∴ For $x(n)$, we have $a_0 = a_1 = a_5 =$

$$a_2 = a_4 = \frac{1}{2}, \quad a_3 = 1$$

$$7.2(a) \quad a_k = 1 + \frac{1}{2} \cos 2\Omega_0 k + 2 \cos \Omega_0 k$$

$$= 1 + \frac{1}{4} e^{-j 2\Omega_0 k} + \frac{1}{4} e^{-j 6\Omega_0 k} + e^{-j \Omega_0 k} + e^{-j 7\Omega_0 k}$$

$$\therefore x(0) = 8, \quad x(1) = x(7) = 8, \quad x(2) = x(6) = 2$$

$$x(3) = x(4) = x(5) = 0$$

$$b) \quad x(n) = e^{j\frac{2\pi}{8}n} + e^{j\frac{2\pi}{8} \cdot 2n} + e^{j\frac{2\pi}{8} \cdot 3n} \\ = e^{j\frac{\pi}{4}n} + e^{j\frac{\pi}{2}n} + e^{j\frac{3\pi}{4}n}$$

$$\therefore x(0) = 3, x(1) = j2.4142, x(2) = -1$$

$$x(3) = -j1.4142, x(4) = -1, x(5) = -j0.4142$$

$$x(6) = -1, x(7) = -j2.4142$$

$$c) \quad x(n) = \sum_{k=0}^7 e^{-j\frac{\pi k}{4}} e^{j\frac{2\pi}{8}kn} = \sum_{k=0}^7 e^{j\frac{\pi}{4}(n-1)k}$$

$$= \begin{cases} \frac{1 - e^{j2\pi(n-1)}}{1 - e^{j\frac{\pi}{4}(n-1)}} & n \neq 1 \end{cases}$$

$$= \begin{cases} 0 & n \neq 1 \\ 1 & n = 1 \end{cases}$$

$$d) \quad x(n) = 1 - e^{-j\frac{4\pi}{5}n} + e^{-j\frac{8\pi}{5}n}$$

$$x(0) = 1, x(1) = 2.118 + j1.5388, \\ = x^*(4)$$

$$x(2) = -0.118 - j0.3633 = x^*(3)$$

$$7.3 (i) \quad b_k = \frac{1}{N} \sum_{\langle n \rangle} x(n-n_0) e^{-jk\Omega_0 n} = \frac{1}{N} \sum_{\langle n \rangle} x(m) e^{-jk\Omega_0(n+m)} \\ = a_k e^{-jk\Omega_0 n_0}$$

$$(ii) \quad b_k = \frac{1}{N} \sum_{\langle n \rangle} x(-n) e^{-jk\Omega_0 n} = \frac{1}{N} \sum_{\langle n \rangle} x(m) e^{-j(-k)\Omega_0 m} = a_{-k} = a_{N-k}$$

(iii) Let N be even. Then $x(-n)$ is periodic with period N

$$\begin{aligned} b_k &= \frac{1}{N} \sum_{\langle n \rangle} (-1)^n x(n) e^{-jk \frac{2\pi}{N} n} = \frac{1}{N} \sum_{\langle n \rangle} x(n) e^{j\pi n} e^{-jk \frac{2\pi}{N} n} \\ &= \frac{1}{N} \sum_{\langle n \rangle} x(n) e^{-j(k - \frac{N}{2}) \frac{2\pi}{N} n} = a_{k - \frac{N}{2}} \end{aligned}$$

If N is odd, $x(-n)$ is periodic with period $2N$.

$$\begin{aligned} b_k &= \frac{1}{2N} \sum_{n=0}^{2N-1} (-1)^n x(n) e^{-jk \frac{\pi}{N} n} \\ &= \frac{1}{2N} \left[\sum_{n=0}^{N-1} x(n) e^{-j(\frac{k-N}{2}) \frac{2\pi}{N} n} + \sum_{n=N}^{2N-1} x(n) e^{-j(\frac{k-N}{2}) \frac{2\pi}{N} n} \right] \\ &= \frac{1}{2} \left[a_{(\frac{k-N}{2})} + \sum_{m=0}^{N-1} x(m+N) e^{-j(\frac{k-N}{2}) \frac{2\pi}{N} (m+N)} \right] \\ &= \frac{1}{2} \left[a_{(\frac{k-N}{2})} + e^{-j(k-N)\pi} a_{(\frac{k-N}{2})} \right] \end{aligned}$$

Where the last step follows since $x(m)$ is periodic with period N . Since N is odd, $k-N$ is odd if k is even and $k-N$ is even if k is odd.

$$\therefore b_k = \begin{cases} a_{(\frac{k-N}{2})} & \text{if } k \text{ is odd} \\ 0 & \text{if } k \text{ is even} \end{cases}$$

$$(iv) \quad y(n) = \frac{1}{2} x(n) + \frac{1}{2} (-1)^n x(n)$$

$$\therefore b_k = \frac{1}{2} a_k + \frac{1}{2} a_{k - \frac{N}{2}} \quad \text{if } N \text{ is even}$$

$$b_k = \begin{cases} \frac{1}{2} a_k + \frac{1}{2} a_{(\frac{k-N}{2})} & \text{for } k \text{ odd, } N \text{ odd} \\ 0 & \text{for } k \text{ even, } N \text{ odd} \end{cases}$$

$$(v) \quad y(n) = \frac{1}{2} x(n) - \frac{1}{2} (-1)^n x(n)$$

$$b_k = \begin{cases} \frac{1}{2} a_k - \frac{1}{2} a_{k - \frac{N}{2}} & N \text{ even} \\ \frac{1}{2} a_k - \frac{1}{2} a_{(\frac{k-N}{2})} & N \text{ odd, } k \text{ odd} \\ 0 & N \text{ odd, } k \text{ even} \end{cases}$$

$$(vi) \quad y(n) = \frac{1}{2} x(n) + \frac{1}{2} x(-n)$$

$$b_k = \frac{1}{2} a_k + \frac{1}{2} a_{N-k}$$

$$(vii) \quad y(n) = \frac{1}{2} x(n) - \frac{1}{2} x(-n)$$

$$b_k = \frac{1}{2} a_k - \frac{1}{2} a_{N-k}$$

$$\begin{aligned} 7.4 \quad a_{N-k} &= \frac{1}{N} \sum_{\langle n \rangle} x(n) e^{-j\Omega_0(N-k)n} \\ &= \frac{1}{N} \sum_{\langle n \rangle} x(n) e^{j\Omega_0 kn} = \frac{1}{N} \left[\sum_{\langle n \rangle} x(n) e^{-j\Omega_0 kn} \right]^* \\ &= a_k^* \end{aligned}$$

$$7.5 \quad H(\Omega) = \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n e^{-j\Omega n} = \frac{1}{1 - \frac{1}{2} e^{-j\Omega}}$$

$$\text{So that } b_k = H(k\Omega_0) a_k = \frac{1}{1 - \frac{1}{2} e^{-jk\Omega_0}} a_k$$

$$7.6 \quad H(\Omega) = \sum_{n=-\infty}^{\infty} \left(\frac{1}{2}\right)^{|n|} e^{-j\Omega n} = \frac{3/4}{1 - \cos \Omega + \frac{1}{4}}$$

$$\text{and } b_k = \frac{3/4}{1 - \cos k\Omega_0 + \frac{1}{4}} a_k$$

$$\begin{aligned} 7.7 \quad (i) \quad C_k &= \frac{1}{N} \sum_n x(n) h(n) e^{-j\Omega_0 kn} \\ &= \frac{1}{N} \sum_n x(n) \left[\sum_m b_m e^{j\Omega_0 mn} \right] e^{-j\Omega_0 kn} \\ &= \sum_m \left[\frac{1}{N} \sum_n x(n) e^{-j\Omega_0 (k-m)n} \right] b_m = \sum_{m=0}^{N-1} a_{k-m} b_m \\ &= a_k \otimes b_k \end{aligned}$$

$$(ii) \quad y(n) = \sum_{m=0}^{N-1} x(m) h(n-m)$$

$$\begin{aligned} C_k &= \frac{1}{N} \sum_{n=0}^{N-1} \left[\sum_{m=0}^{N-1} x(m) h(n-m) \right] e^{-j\Omega_0 kn} \\ &= \left[\frac{1}{N} \sum_{m=0}^{N-1} x(m) e^{-j\Omega_0 km} \right] \left[\sum_{n=0}^{N-1} h(n-m) e^{-j\Omega_0 k(n-m)} \right] \\ &= N a_k b_k \end{aligned}$$

$$7.8(a) \quad a_k = [1 + e^{-j\frac{2\pi}{3}k} + e^{-j\frac{4\pi}{3}k}] / 6$$

$$b_k = [1 - e^{-j\frac{\pi}{3}k} + e^{-j\frac{2\pi}{3}k} - e^{-j\frac{4\pi}{3}k} + e^{-j\frac{5\pi}{3}k}]$$

$$a_0 = \frac{1}{2}, a_1 = 0.1667, a_2 = j \cdot 0.2887, a_3 = 0.1667$$

$$a_4 = -j \cdot 0.2887, a_5 = 0.1667$$

$$b_0 = 0.1667, b_1 = 0.1667, b_2 = 0.1667 + j \cdot 0.5774$$

$$b_3 = 0.1667, b_4 = 0.1667 - j \cdot 0.5774, b_5 = 0.1667$$

$$a \quad a_0 \quad a_1 \quad a_2 \quad a_3 \quad a_4 \quad a_5$$

$$a_{11}$$

$$C_0 = 0.5, C_1 = 0.1667, C_2 = j \cdot 0.2887 = C_4^*$$

$$C_3 = 0.1667 = C_5$$

$$b) \quad x(n) = \frac{1}{2} e^{j\Omega_0 n} + \frac{1}{2} e^{-j\Omega_0 n} \quad \text{where } \Omega_0 = \frac{2\pi}{6} = \frac{\pi}{3}$$

$$\therefore a_0 = 0, a_1 = a_5 = \frac{1}{2}, a_2 = a_3 = a_4 = 0$$

$$b_k = \frac{1}{6} [1 - e^{-j\frac{\pi}{3}k} + e^{-j\frac{2\pi}{3}k} + e^{-j\pi k} - e^{-j\frac{4\pi}{3}k} + e^{-j\frac{5\pi}{3}k}]$$

$$b_0 = \frac{1}{3}, b_1 = 0, b_2 = 0.3333 + j \cdot 0.5774, b_3 = 0$$

$$= b_4^*$$

$$C_0 = 0, C_1 = 0.1111 + j \cdot 0.0962, C_2 = 0, C_3 = -0.1111$$

$$C_4 = 0, C_5 = 0.1111 - j \cdot 0.0962$$

$$c) \quad x(n) = e^{j\Omega_0 n} + e^{-j\Omega_0 n} \quad \text{where } \Omega_0 = \pi/2$$

$$\therefore a_0 = 0, a_1 = 1, a_2 = 0, a_3 = -1$$

$$b_k = 1 - \frac{1}{2} e^{-j\pi/2 k} + \frac{1}{4} e^{-j\pi k} - \frac{1}{8} e^{-j3\pi/2 k}$$

$$b_0 = .1562, b_1 = .1875 + j0.0938 = b_3^*$$

$$b_2 = .4688$$

$$c_0 = .375 = c_2, c_1 = .625 = c_3$$

$$d) \quad a_k = \frac{1}{8} \sum_{n=0}^7 e^{-j2\pi/8 n k} = \begin{cases} 0 & k \neq 0 \\ 1 & k = 0 \end{cases}$$

$$b_k = \frac{1}{8} \left[1 + 2e^{-j\pi/4 k} + 3e^{-j\pi/2 k} + 4e^{-j3\pi/4 k} + 4e^{-j\pi k} + 3e^{-j5\pi/4 k} + 2e^{-j3\pi/2 k} + e^{-j7\pi/4 k} \right]$$

$$b_0 = 2.5, b_1 = -.7286 - .3018j = b_7^*$$

$$b_2 = 0 = b_6$$

$$b_3 = -.0214 - .0518j, b_4 = 0 \\ = b_5^*$$

$$c_k = b_k$$

$$7.9 \quad c_k = N a_k b_k \quad \text{where } a_k \text{ and } b_k \text{ are as in 7.8}$$

$$(i) \quad c_k = \{0, 0, 3 - j3\sqrt{3}, 0, 3 + j3\sqrt{3}, 0\}$$

$$(ii) \quad c_0 = \frac{119}{8}, \quad c_1 = \frac{1}{8\sqrt{2}} [(1+\sqrt{2}) + j(1-\sqrt{2})]$$

$$c_2 = \frac{1}{8\sqrt{2}} [(-1+\sqrt{2}) + j(1-\sqrt{2})]$$

$$c_3 = \frac{1}{8}, \quad c_4 = \frac{1}{8\sqrt{2}} [(-1+\sqrt{2}) - j(1-\sqrt{2})]$$

$$c_5 = \frac{1}{8\sqrt{2}} [(1+\sqrt{2}) - j(1-\sqrt{2})]$$

$$(iii) \quad r = 0, \quad \phi = 0, 1, 5$$

$$(iv) \quad c_k = [0, 0, \frac{15}{8}, 0]$$

$$7.10 \quad \frac{1}{2\pi} \int_0^{2\pi} e^{j\Omega(n-k)} d\Omega = \begin{cases} 1 & n=k \\ \frac{e^{j2\pi(n-k)} - 1}{j2\pi(n-k)} & n \neq k \end{cases} = \delta(n-k)$$

$$7.11 \quad X(\Omega) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\Omega n}$$

$$\frac{dX(\Omega)}{d\Omega} = \sum_{n=-\infty}^{\infty} x(n) (-jn) e^{-j\Omega n}$$

$$\therefore \mathcal{F}\{nx(n)\} = j \frac{dX(\Omega)}{d\Omega}$$

Repeat p times to get the desired result.

$$7.12 (i) \quad X(\Omega) = \sum_{n=0}^{n_0} e^{-j\Omega n} = \frac{1 - e^{-j\Omega(n_0+1)}}{1 - e^{-j\Omega}}$$

$$= \frac{\sin \frac{\Omega(n_0+1)}{2}}{\sin \frac{\Omega}{2}} e^{-j\Omega \frac{n_0}{2}}$$

$$\begin{aligned}
 \text{(ii)} \quad \mathcal{F} \left\{ \left(\frac{1}{3} \right)^{|n|} \right\} &= \sum_{n=-\infty}^{-1} \left(\frac{1}{3} \right)^{-n} e^{-j\Omega n} + \sum_{n=0}^{\infty} \left(\frac{1}{3} \right)^n e^{-j\Omega n} \\
 &= \frac{\frac{1}{3} e^{j\Omega}}{1 - \frac{1}{3} e^{j\Omega}} + \frac{1}{1 - \frac{1}{3} e^{-j\Omega}} = \frac{4}{5 - 3 \cos \Omega}
 \end{aligned}$$

$$\begin{aligned}
 \mathcal{F} \left\{ n \left(\frac{1}{3} \right)^{|n|} \right\} &= j \frac{d}{d\Omega} \frac{4}{5 - 3 \cos \Omega} \\
 &= j \frac{3 \sin \Omega}{(5 - 3 \cos \Omega)^2}
 \end{aligned}$$

$$\text{(iii)} \quad \mathcal{F} \{ a^n a(n) \} = \frac{1}{1 - 2e^{-j\Omega}}$$

$$\begin{aligned}
 \therefore \mathcal{F} \{ a^n \cos \Omega_0 n u(n) \} &= \mathcal{F} \left\{ \frac{1}{2} a^n e^{j\Omega_0 n} u(n) + \frac{1}{2} a^n e^{-j\Omega_0 n} u(n) \right\} \\
 &= \frac{1}{2} \frac{1}{1 - a e^{-j(\Omega - \Omega_0)}} + \frac{1}{2} \frac{1}{1 - a e^{-j(\Omega + \Omega_0)}} \\
 &= \frac{1 - a \cos \Omega_0 e^{-j\Omega}}{1 - 2a e^{-j\Omega} \cos \Omega_0 + a^2 e^{-j2\Omega}}
 \end{aligned}$$

$$\text{(iv)} \quad \mathcal{F} \{ e^{j3n} \} = \sum_{k=-\infty}^{\infty} 2\pi \delta(\Omega - 3 - 2\pi k)$$

$$\text{(v)} \quad \mathcal{F} \{ e^{j\frac{\pi}{8}n} \} = \sum_{k=-\infty}^{\infty} 2\pi \delta(\Omega - k\frac{\pi}{8})$$

$$\begin{aligned}
 \text{(vi)} \quad x(n) &= \frac{1}{4j} e^{j3n} - \frac{1}{4j} e^{-j3n} + 2 e^{j\frac{\pi}{8}n} + 2 e^{-j\frac{\pi}{8}n} \\
 X(\Omega) &= \frac{1}{4j} \sum_{k=-\infty}^{\infty} 2\pi \delta(\Omega - 3 - 2\pi k) - \frac{1}{4j} \sum_{k=-\infty}^{\infty} 2\pi \delta(\Omega + 3 - 2\pi k) \\
 &\quad + 4 \sum_{k=-\infty}^{\infty} 2\pi \delta(\Omega - k\frac{\pi}{8})
 \end{aligned}$$

$$(vii) \quad X(\Omega) = \sum_{n=0}^{n_0-1} a^n e^{-j\Omega n} = \frac{1 - (ae^{-j\Omega})^{n_0}}{1 - ae^{-j\Omega}}$$

$$(viii) \quad X(\Omega) = \text{rect}\left(\frac{\Omega}{\frac{1}{3}}\right)$$

$$(ix) \quad X(\Omega) = \frac{1}{2\pi} \text{rect}\left(\frac{\Omega}{\frac{1}{3}}\right) * \text{rect}\left(\frac{\Omega}{\frac{1}{2}}\right)$$

$$= \begin{cases} \Omega + \frac{5}{12} & -\frac{5}{12} \leq \Omega < -\frac{1}{12} \\ \frac{1}{3} & -\frac{1}{12} \leq \Omega < \frac{1}{12} \\ -\Omega + \frac{5}{12} & \frac{1}{12} \leq \Omega < \frac{5}{12} \\ 0 & \text{otherwise} \end{cases}$$

$$(x) \quad X(\Omega) = \frac{1}{2j} \text{rect}\left(\frac{\Omega - \pi/2}{\frac{1}{3}}\right) - \frac{1}{2j} \text{rect}\left(\frac{\Omega + \pi/2}{\frac{1}{3}}\right)$$

$$(xi) \quad X(\Omega) = \frac{1}{1 - ae^{-j\Omega}} + \frac{ae^{-j\Omega}}{(1 - ae^{-j\Omega})^2} = \frac{1}{(1 - ae^{-j\Omega})^2}$$

$$7.13 (a) \quad x(n) = \frac{-j}{2} e^{j\frac{\pi}{3}n} + \frac{1}{2} e^{j\frac{2\pi}{3}n} + \frac{1}{2} e^{j\frac{4\pi}{3}n} + \frac{j}{2} e^{j\frac{5\pi}{3}n}$$

$$= \frac{1}{2} \cos \frac{2\pi n}{3} + \frac{1}{2} \sin \frac{\pi n}{3}$$

$$b) \quad X(\Omega) = \frac{4(e^{j5\Omega} - 4e^{-j5\Omega})}{2j} + \frac{2(e^{j3\Omega} + e^{-j3\Omega})}{2}$$

$$= -j2\delta(n+5) + j2\delta(n-5) + \delta(n-6) + \delta(n+3)$$

$$\begin{aligned}
 c) \quad X(\Omega) &= \frac{1}{\left[1 - \frac{1}{2}e^{-j\Omega}\right]^2} = 2e^{j\Omega} \times \frac{\frac{1}{2}e^{-j\Omega}}{\left[1 - \frac{1}{2}e^{-j\Omega}\right]^2} \\
 &= 2(n+1) \left(\frac{1}{2}\right)^{n+1} u(n+1)
 \end{aligned}$$

$$\begin{aligned}
 d) \quad X(\Omega) &= \frac{2}{1 + \frac{1}{3}e^{-j\Omega}} + \frac{-1}{1 - \frac{1}{4}e^{-j\Omega}} \\
 x(n) &= 2\left(-\frac{1}{3}\right)^n u(n) - \left(\frac{1}{4}\right)^n u(n)
 \end{aligned}$$

$$\begin{aligned}
 7.14 \quad \sum_{n=-\infty}^{\infty} |x(n)|^2 &= \sum_{n=-\infty}^{\infty} x(n) \frac{1}{2\pi} \int_{-\infty}^{\infty} X^*(\Omega) e^{j\Omega n} d\Omega \\
 &= \frac{1}{2\pi} \int_{-\infty}^{\infty} X^*(\Omega) \left[\sum_{n=-\infty}^{\infty} x(n) e^{j\Omega n} \right] d\Omega \\
 &= \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(\Omega)|^2 d\Omega
 \end{aligned}$$

$$7.15 \quad H(\Omega) = \frac{1}{1 - \frac{1}{2}e^{-j\Omega}}$$

$$(i) \quad X(\Omega) = \frac{1 - \frac{1}{3}\cos\frac{\pi}{2}e^{-j\Omega}}{1 - \frac{2}{3}\cos\frac{\pi}{2}e^{-j\Omega} + \frac{1}{9}e^{-j2\Omega}} = \frac{1}{1 + \frac{1}{9}e^{-j2\Omega}}$$

$$\begin{aligned}
 \therefore Y(\Omega) &= H(\Omega) X(\Omega) = \frac{1}{(1 + \frac{1}{9}e^{-j2\Omega})(1 - \frac{1}{2}e^{-j\Omega})} \\
 &= \frac{9/13}{1 - \frac{1}{2}e^{-j\Omega}} + \frac{\frac{2}{13}e^{-j\Omega} + \frac{4}{13}}{1 + \frac{1}{9}e^{-j2\Omega}}
 \end{aligned}$$

$$= \frac{9/13}{1 - \frac{1}{2} e^{-j\Omega}} + \frac{4/13}{1 + \frac{1}{9} e^{-j2\Omega}} + \frac{6}{13} \frac{1/3}{1 + \frac{1}{9} e^{-j2\Omega}}$$

$$\therefore y(n) = \frac{9}{13} \left(\frac{1}{2}\right)^n + \frac{4}{13} \left(\frac{1}{3}\right)^n \cos \frac{\pi}{2} n + \frac{6}{13} \left(\frac{1}{3}\right)^n \sin \left(\frac{\pi}{2}\right)^n n \geq 0$$

where we have used

$$\mathcal{Z} \left\{ a^n \cos \Omega_0 n u(n) \right\} = \frac{1 - a \cos \Omega_0 e^{-j\Omega}}{1 - 2a \cos \Omega_0 e^{-j\Omega} + a^2 e^{-j2\Omega}}$$

$$\mathcal{Z} \left\{ a^n \sin \Omega_0 n u(n) \right\} = \frac{a \sin \Omega_0 e^{-j\Omega}}{1 - 2a \cos \Omega_0 e^{-j\Omega} + a^2 e^{-j2\Omega}}$$

$$(ii) X(\Omega) = \frac{\frac{1}{3} e^{-j\Omega}}{1 + \frac{1}{9} e^{-j2\Omega}}$$

$$\begin{aligned} Y(\Omega) &= \frac{\frac{1}{3} e^{-j\Omega}}{(1 + \frac{1}{9} e^{-j2\Omega})(1 - \frac{1}{2} e^{-j\Omega})} \\ &= \frac{6/13}{1 - \frac{1}{2} e^{-j\Omega}} + \frac{4/39 e^{-j\Omega} - \frac{6}{13}}{1 + \frac{1}{9} e^{-j2\Omega}} \end{aligned}$$

$$\therefore y(n) = \frac{6}{13} \left(\frac{1}{2}\right)^n + \frac{12}{39} \left(\frac{1}{3}\right)^n \sin \frac{\pi}{2} n - \frac{6}{13} \left(\frac{1}{3}\right)^n \cos \frac{n\pi}{2} \quad \text{for } n \geq 0$$

$$(iii) X(\Omega) = \frac{8/9}{1 - \frac{2}{3} (\cos \Omega + \frac{1}{9})} = \frac{8/9}{(1 - \frac{1}{3} e^{-j\Omega})(1 - \frac{1}{3} e^{j\Omega})}$$

$$\begin{aligned} Y(\Omega) &= \frac{8/9}{(1 - \frac{1}{3} e^{-j\Omega})(1 - \frac{1}{3} e^{j\Omega})} \cdot \frac{1}{1 - \frac{1}{2} e^{-j\Omega}} \\ &= \frac{-2}{1 - \frac{1}{3} e^{-j\Omega}} + \frac{2/5 e^{j\Omega}}{1 - \frac{1}{3} e^{j\Omega}} + \frac{16/5}{1 - \frac{1}{2} e^{-j\Omega}} \end{aligned}$$

$$y(n) = \begin{cases} 6/5 (3)^n & n < 0 \\ -2 \left(\frac{1}{3}\right)^n + \frac{16}{5} \left(\frac{1}{2}\right)^n & n \geq 0 \end{cases}$$

$$(iv) X(\omega) = \frac{\frac{1}{3} e^{-j\omega}}{(1 - \frac{1}{3} e^{-j\omega})^2}$$

$$Y(\omega) = \frac{\frac{1}{3} e^{-j\omega}}{(1 - \frac{1}{3} e^{-j\omega})^2 (1 - \frac{1}{2} e^{-j\omega})}$$

$$= \frac{6}{1 - \frac{1}{2} e^{-j\omega}} + \frac{-2}{(1 - \frac{1}{3} e^{-j\omega})^2} - \frac{4}{1 - \frac{1}{3} e^{-j\omega}}$$

$$y(n) = 6 \left(\frac{1}{2}\right)^n u(n) - 2(n+1) \left(\frac{1}{3}\right)^{n+1} u(n+1) - 4 \left(\frac{1}{3}\right)^n u(n)$$

$$= 6 \left(\frac{1}{2}\right)^n u(n) - \frac{2}{3} n \left(\frac{1}{3}\right)^n u(n) - \frac{14}{3} \left(\frac{1}{3}\right)^n u(n)$$

$$7.16 (a) X(\omega) = \frac{1}{1 + \frac{1}{9} e^{-j2\omega}}$$

$$Y(\omega) = \frac{e^{-j\omega}}{(1 + \frac{1}{9} e^{-j2\omega})} + \frac{1}{(1 + \frac{1}{9} e^{-j2\omega}) (1 - \frac{1}{4} e^{-j\omega})}$$

$$= \frac{e^{-j\omega}}{(1 + \frac{1}{9} e^{-j2\omega})} + \frac{\frac{4}{25} e^{-j\omega} + \frac{16}{25}}{1 + \frac{1}{9} e^{-j2\omega}} + \frac{9/25}{1 - \frac{1}{4} e^{-j\omega}}$$

$$= \frac{9/25}{1 - \frac{1}{4} e^{-j\omega}} + \frac{e^{-j\omega}}{1 + \frac{1}{9} e^{-j2\omega}} + \frac{16/25}{1 + \frac{1}{9} e^{-j2\omega}} + \frac{12}{25} \frac{1/3}{1 + \frac{1}{9} e^{-j2\omega}}$$

$$y(n) = \frac{9}{25} \left(\frac{1}{4}\right)^n u(n) + \left(\frac{1}{3}\right)^{n-1} \sin \frac{\pi n}{2} u(n-1) +$$

$$\left[\frac{16}{25} \left(\frac{1}{9}\right)^n \cos \frac{\pi n}{2} + \frac{12}{25} \left(\frac{1}{3}\right)^n \sin \frac{\pi n}{2} \right] u(n)$$

$$\text{iii)} \quad X(z) = \frac{8/9}{(1 - \frac{1}{3} e^{-jz})(1 - \frac{1}{3} e^{jz})}$$

$$Y(z) = \frac{8/9 e^{-jz}}{1 - \frac{2}{3} \cos z + \frac{1}{9}} + \frac{2/9}{(1 - \frac{1}{3} e^{-jz})(1 - \frac{1}{3} e^{jz})(1 - \frac{1}{4} e^{jz})}$$

$$= \frac{8/9 e^{-jz}}{1 - \frac{2}{3} \cos z + \frac{1}{9}} + \frac{4}{1 - \frac{1}{3} e^{-jz}} - \frac{12/11 e^{jz}}{1 - \frac{1}{3} e^{jz}}$$

$$- \frac{32/11}{1 - \frac{1}{4} e^{-jz}}$$

$$y(n) = \frac{8}{9} \left[\frac{1}{3} \left(\frac{1}{3}\right)^{-n} u(-n-1) + \frac{1}{3} \delta(n) + 3 \left(\frac{1}{3}\right)^n u(n-1) \right]$$

$$+ 4 \left(\frac{1}{3}\right)^n u(n) - \frac{12}{11} \left(\frac{1}{3}\right)^{-n} u(-n-1)$$

$$- \frac{32}{11} \left(\frac{1}{4}\right)^n u(n)$$

$$\text{d)} \quad X(z) = \frac{\frac{1}{3} e^{-jz}}{(1 - \frac{1}{3} e^{-jz})^2}$$

$$Y(z) = \frac{\frac{1}{3} e^{-j2z}}{(1 - \frac{1}{3} e^{-jz})^2} + \frac{\frac{1}{3} e^{-jz}}{(1 - \frac{1}{3} e^{-jz})^2 (1 - \frac{1}{4} e^{-jz})}$$

$$= \frac{\frac{1}{3} e^{-j2z}}{(1 - \frac{1}{3} e^{-jz})^2} + \frac{4/3}{(1 - \frac{1}{3} e^{-jz})^2} + \frac{-12}{1 - \frac{1}{3} e^{-jz}} + \frac{12}{1 - \frac{1}{4} e^{-jz}}$$

$$y(n) = (n-1) \left(\frac{1}{3}\right)^{n-1} u(n-1) + 4/3 (n+1) \left(\frac{1}{3}\right)^{n+1} u(n+1)$$

$$- 12 \left(\frac{1}{3}\right)^n u(n) + 12 \left(\frac{1}{4}\right)^n u(n)$$

$$ii) X(\omega) = \frac{1/3 e^{-j\omega}}{1 + \frac{1}{9} e^{-j2\omega}}$$

$$Y(\omega) = \frac{1/3 e^{-j2\omega}}{1 + 1/9 e^{-j2\omega}} + \frac{1/3 e^{-j\omega}}{(1 + 1/9 e^{-j2\omega})(1 - 1/4 e^{-j\omega})}$$

$$= \frac{1/3 e^{-j2\omega}}{1 + \frac{1}{9} e^{-j2\omega}} + \frac{-12/25 + \frac{16}{75} e^{-j\omega}}{1 + 1/9 e^{-j2\omega}} + \frac{12/25}{1 - 1/4 e^{-j\omega}}$$

$$y(n) = \left(\frac{1}{3}\right)^{n-1} \sin \frac{\pi(n-1)}{2} u(n-1) - \frac{12}{25} \left(\frac{1}{3}\right)^n \cos \frac{\pi n}{2} u(n) + \frac{48}{75} \left(\frac{1}{3}\right)^n \sin \frac{\pi n}{2} u(n) + \frac{12}{25} \left(\frac{1}{4}\right)^n u(n)$$

$$7.17 H(\omega) = \text{rect} \left(\frac{\omega}{\pi/2} \right)$$

(i) The Fourier Series Coefficients are

$$a_0 = \frac{2}{3}$$

$$a_k = \frac{1}{6} \sum_{n=0}^3 e^{-j \frac{2\pi}{6} n k} = \frac{1}{6} \frac{1 - e^{-jk \frac{4\pi}{3}}}{1 - e^{-jk \frac{\pi}{3}}} \quad k \neq 0$$

$$\text{i.e. } a_k = \frac{1}{6} e^{-jk\pi/2} \frac{\sin 2\pi k/3}{\sin k\pi/6} \quad \text{for } k \neq 0$$

Filter only passes frequencies in the range

$|\omega| < \frac{\pi}{4}$ so that only dc term is passed through.

Thus output spectrum is

$$Y(\omega) = 2\pi \sum_{m=-\infty}^{\infty} \frac{2}{3} \delta(\omega - 2\pi m)$$

$$\text{and } y(n) = \frac{2}{3}$$

(ii) $x(n)$ is periodic with $N=2$ and $x(0)=1$ $x(1)=-1$

$$\therefore a_0 = 0$$

$$\therefore a_1 = 1$$

$$\therefore X(\Omega) = 2\pi \sum_{m=-\infty}^{\infty} \delta(\Omega - \pi - 2\pi m)$$

so that $Y(\Omega) = 0$ and $y(n) = 0$

7.18 Since $H(\Omega) = \text{rect}\left(\frac{\Omega}{3\pi/4}\right)$, in case (i),

only dc & 1st harmonic are passed through

$$\therefore Y(\Omega) = 2\pi \sum_{m=-\infty}^{\infty} \left[\frac{2}{3} \delta(\Omega - 2\pi m) + \frac{\sqrt{3}}{6j} \delta(\Omega - \frac{\pi}{3} - 2\pi m) - \frac{\sqrt{3}}{6j} \delta(\Omega - \frac{5\pi}{3} - 2\pi m) \right]$$

$$\text{and } y(n) = \frac{2}{3} + \frac{1}{\sqrt{3}} \sin \frac{\pi n}{3}$$

For case (ii), again $Y(\Omega) = 0$ so that $y(n) = 0$

$$7.19 (i) \left[1 + \frac{5}{8} e^{-j\Omega} + \frac{3}{22} e^{-j2\Omega} \right] Y(\Omega) = [1 - e^{-j\Omega}] X(\Omega)$$

$$\therefore H(\Omega) = \frac{1 - e^{-j\Omega}}{1 + \frac{5}{8} e^{-j\Omega} + \frac{3}{32} e^{-j2\Omega}}$$

$$= \frac{11}{1 + \frac{3}{8} e^{-j\Omega}} - \frac{10}{1 + \frac{1}{4} e^{-j\Omega}}$$

$$h(n) = 11 \left(-\frac{3}{8}\right)^n u(n) - 10 \left(-\frac{1}{4}\right)^n u(n)$$

$$(ii) \left[1 + e^{-j\Omega} + \frac{1}{4} e^{-j2\Omega} \right] Y(\Omega) = X(\Omega)$$

$$H(\Omega) = \frac{1}{1 + e^{-j\Omega} + \frac{1}{8} e^{-j2\Omega}}$$

$$= \frac{\frac{1-\sqrt{2}}{2}}{\left[1 + \left(\frac{1}{2} - \frac{j}{2\sqrt{2}}\right) e^{-j\Omega} \right]} + \frac{\frac{1+\sqrt{2}}{2}}{\left[1 + \left(\frac{1}{2} + \frac{j}{2\sqrt{2}}\right) e^{-j\Omega} \right]}$$

$$h(n) = \left(\frac{1-\sqrt{2}}{2}\right) \left[-\frac{1}{2} + \frac{1}{2\sqrt{2}}\right]^n u(n) + \left(\frac{1+\sqrt{2}}{2}\right) \left(-\frac{1}{2} - \frac{1}{2\sqrt{2}}\right)^n u(n)$$

$$(iv) [1 - 3e^{-j\Omega} + 2e^{-j2\Omega}] Y(\Omega) = X(\Omega)$$

$$H(\Omega) = \frac{1}{1 - 3e^{-j\Omega} + 2e^{-j2\Omega}}$$

This is not a valid Fourier transform and so

$H(\Omega)$ does not exist for this system

$$(v) [e^{j2\Omega} - \frac{1}{12}e^{j\Omega} - \frac{1}{12}] Y(\Omega) = [1 + \frac{1}{2}e^{j\Omega}] X(\Omega)$$

$$H(\Omega) = \frac{1 + \frac{1}{2}e^{j\Omega}}{e^{j2\Omega} - \frac{1}{12}e^{j\Omega} - \frac{1}{12}}$$

$$= \frac{-\frac{3}{2}}{1 + \frac{1}{4}e^{-j\Omega}} + \frac{2}{1 - \frac{1}{2}e^{-j\Omega}}$$

$$y(n) = -\frac{3}{2} \left(-\frac{1}{4}\right)^n u(n) + 2 \left(\frac{1}{2}\right)^n u(n)$$

7-20

$$H(\Omega) = \frac{\frac{1}{2} + \frac{1}{12}e^{-j\Omega}}{1 + \frac{5}{6}e^{-j\Omega} + \frac{1}{6}e^{-j2\Omega}}$$

$$= \frac{\frac{1}{2}}{1 + \frac{1}{3}e^{-j\Omega}} + \frac{1}{1 + \frac{1}{2}e^{-j\Omega}}$$

$$(a) h(n) = -\frac{1}{2} \left(-\frac{1}{3}\right)^n u(n) + \left(-\frac{1}{2}\right)^n u(n)$$

$$(b) \text{ Let } H(\Omega) = \frac{Y(\Omega)}{X(\Omega)}. \text{ Then}$$

$$\left[1 + \frac{5}{6}e^{-j\Omega} + \frac{1}{6}e^{-j2\Omega}\right] Y(\Omega) = \left[\frac{1}{2} + \frac{1}{12}e^{-j\Omega}\right] X(\Omega)$$

$$\therefore y(n) + \frac{5}{6}y(n-1) + \frac{1}{6}y(n-2) = \frac{1}{2}x(n) + \frac{1}{12}x(n-1)$$

$$(c) \quad X(z) = \frac{1}{1 - \frac{1}{2}e^{-j\Omega}}$$

$$\begin{aligned} Y(z) &= H(z)X(z) = \frac{\frac{1}{2} + \frac{1}{2}e^{-j\Omega}}{(1 + \frac{1}{3}e^{-j\Omega})(1 + \frac{1}{2}e^{-j\Omega})(1 - \frac{1}{2}e^{-j\Omega})} \\ &= \frac{-\frac{1}{5}}{(1 + \frac{1}{3}e^{-j\Omega})} + \frac{\frac{1}{2}}{(1 + \frac{1}{2}e^{-j\Omega})} + \frac{\frac{1}{5}}{(1 - \frac{1}{2}e^{-j\Omega})} \end{aligned}$$

$$\begin{aligned} \therefore y(n) &= -\frac{1}{5}\left(\frac{1}{3}\right)^n u(n) + \frac{1}{2}\left(-\frac{1}{2}\right)^n u(n) \\ &\quad + \frac{1}{5}\left(\frac{1}{2}\right)^n u(n) \end{aligned}$$

$$7.21 \quad |H(z)|^2 = \frac{\alpha + e^{-j\Omega}}{1 + \beta e^{-j\Omega}} \cdot \frac{\alpha^* + e^{j\Omega}}{1 + \beta^* e^{j\Omega}} = \frac{|\alpha|^2 + 1 + \alpha e^{j\Omega} + \alpha^* e^{-j\Omega}}{1 + |\beta|^2 + \beta e^{j\Omega} + \beta^* e^{-j\Omega}}$$

clearly if we choose $\alpha^* = \beta$, $|H(z)|^2 = 1$

More generally, if we choose $\alpha = \frac{1}{\beta}$, we get

$$|H(z)|^2 = \frac{1}{|\beta|^2} = \text{Constant which is also an all pass function}$$

7.22 (a)

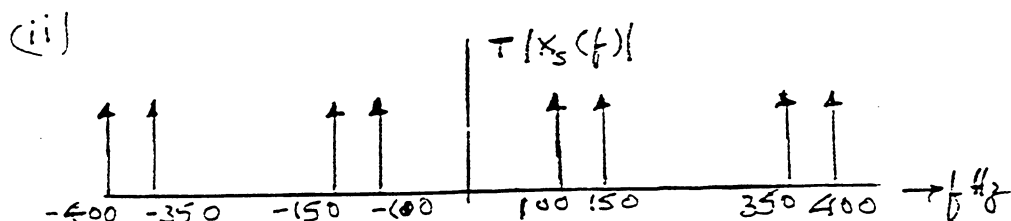
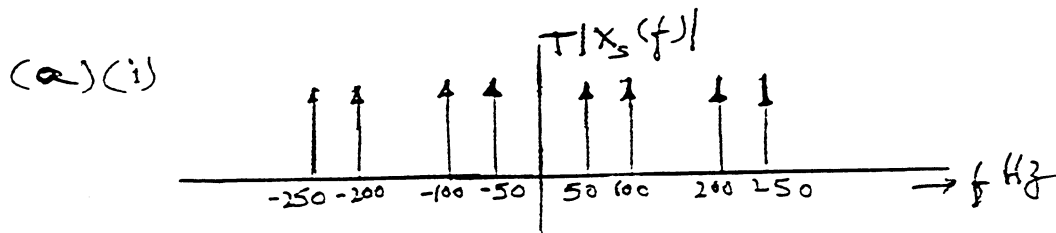
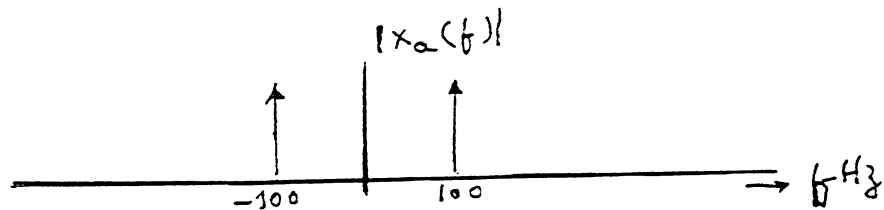
$$|H(z)|^2 = \frac{1 + ae^{-j\Omega} + be^{-j2\Omega}}{b + ae^{-j\Omega} + e^{-j2\Omega}} \cdot \frac{1 + ae^{j\Omega} + be^{j2\Omega}}{b + ae^{j\Omega} + e^{j2\Omega}}$$

= 1

$$(b) \quad \text{Let } D(z) = \sum_{k=0}^N a_k e^{-jk\Omega} \quad \text{with } a_k \text{ real.}$$

$$\text{Then } |H(z)|^2 = 1 \text{ if } N(z) = \sum_{k=0}^N a_{N-k} e^{-jk\Omega}$$

7.23



(b) In both cases, $x_a(t)$ can be recovered by using a band-pass filter centered around 100 Hz and with a bandwidth of say 50 Hz. In case (i), we can also use a low-pass filter with cutoff, say 125 Hz.

7.24 To find the impulse response, let $x_a(0) = 1$, with all other values zero. Then, for $-T \leq t < 0$,

$$g(t) = \hat{x}_0(t) = 1 + \frac{t}{T}$$

In the range $0 \leq t < T$, $g(t) = -\frac{t+T}{T} = 1 - \frac{t}{T}$

$$\therefore g(t) = \begin{cases} 1 - \frac{t}{T} & |t| < T \\ 0 & \text{otherwise} \end{cases}$$

$$= \frac{1}{\sqrt{T}} \text{rect}\left(\frac{t}{T}\right) * \frac{1}{\sqrt{T}} \text{rect}\left(\frac{t}{T}\right)$$

$$\therefore G_1(\omega) = \frac{1}{T} \left[\frac{2 \sin \frac{\omega T}{2}}{\omega} \right]^2 = T \left[\frac{\sin \frac{\omega T}{2}}{\frac{\omega T}{2}} \right]^2$$

7.25 (a) $T = \frac{1}{10,000}$ s. Since the signal is bandlimited to 1.5 kHz, $f_s \geq 3000$ Hz for no aliasing.

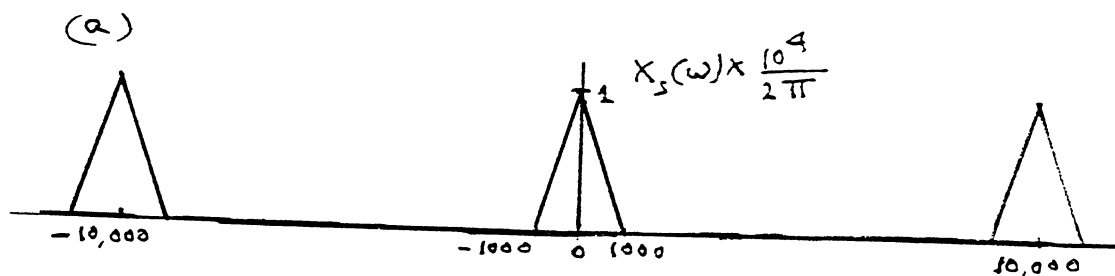
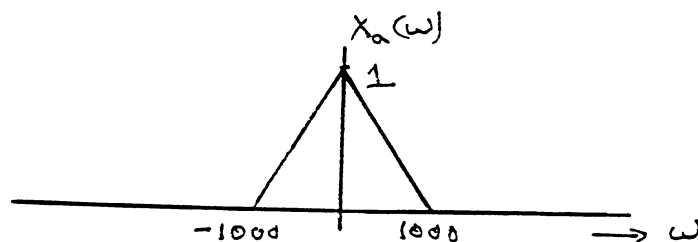
$$\therefore T' = MT < \frac{1}{3000}$$

$$M < \text{Int}\left[\frac{10,000}{3000}\right] = 3$$

(b) $\frac{10,000}{4,000} = \frac{5}{2}$

$\therefore$ First interpolate by a factor of 5 and then decimate by a factor of 2.

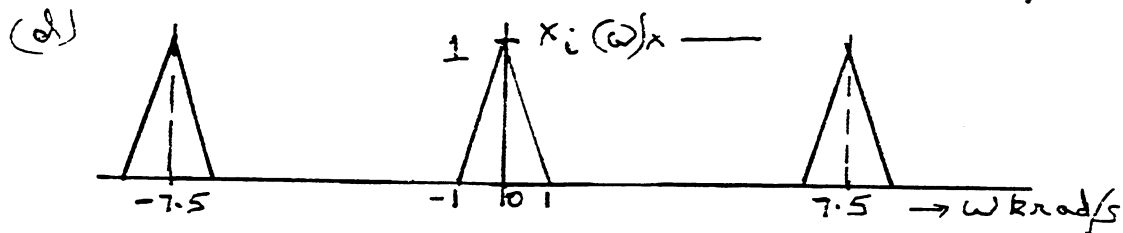
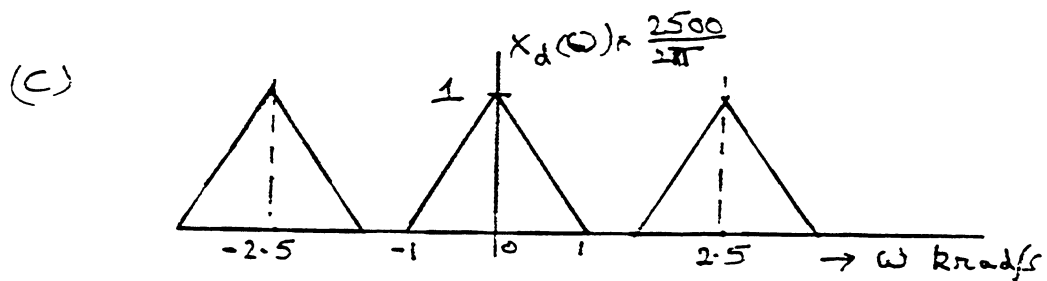
7.26



(b) Largest T' that can be used without aliasing distortion is $T' = \frac{2\pi}{2000} = MT = M \cdot \frac{2\pi}{10,000}$

$$\therefore M = 5$$

(c) $M = 4$ corresponds to $T' = 4 \times \frac{2\pi}{10,000}$ or a sampling frequency of $\omega_s = 2500$ rad/sec



7.27 From Eqn. (7.5.38), we have

$$E(t) = \frac{\Delta}{T_2 - T_1} \left[t - \frac{T_2 + T_1}{2} \right]$$

$$\begin{aligned} E &= \frac{1}{T_2 - T_1} \int_{T_1}^{T_2} \frac{\Delta^2}{(T_2 - T_1)^2} \left[t - \frac{T_2 + T_1}{2} \right]^2 dt \\ &= \frac{\Delta^2}{(T_2 - T_1)^3} \int_{\frac{T_2 - T_1}{2}}^{\frac{T_2 + T_1}{2}} \tau^2 d\tau = \frac{\Delta^2}{12} \end{aligned}$$

7.28 (a) $(SNR)_{dB} = 10.79 + 10 \log_{10} P_x + 6.02B - 20 \log_{10} D$
 $= 10.79 + 10 \log_{10}(10) + 6.02B - 20 \log_{10}(40)$
 $= 5$
 $\therefore B = 2.7$. Use 3 bits.

(b) With $B = 3$ and $D = 2$, we have

$$\begin{aligned} (SNR)_{dB} &= 10.79 + 10 \log_{10}(10) + 6.02 \times 3 - 20 \log_{10}(40) \\ &= 12.83 \text{ dB} \end{aligned}$$

7.29 (a) $10.79 + 10 \log_{10}(10) + 6.02B - 20 \log_{10}(40) = 10$ gives
 $B = 3.53$. Set $B = 4$ bits

(b) With B as in part (a) and $D = 20$, we have

$$(SNR)_{dB} = 10.79 + 10 + 6.02 \times 4 - 20 \log_{10}(20) = 18.85 \text{ dB}$$

Chapter 8

8.1 (a) $X(z) = \sum_{n=-\infty}^{-1} (-3)^n z^{-n} = \frac{z}{z+3} ; |z| < 3$

(b) $X(z) = \sum_{n=-5}^5 z^{-n}$ for $z \neq 0$

(c) $X(z) = \sum_{n=-\infty}^{-1} 3^n z^{-n} + \sum_{n=0}^{\infty} \left(\frac{1}{3}\right)^n z^{-n}$

First term converges for $|z| < 3$, second for $|z| > \frac{1}{3}$.

$\therefore X(z) = \frac{-z}{z-3} + \frac{z}{z-\frac{1}{3}} ; \frac{1}{3} < |z| < 3$

(d) $X(z) = 2 - \sum_{n=0}^{\infty} 2^n z^{-n} = 2 - \frac{z}{z-2} = \frac{z-4}{z-2} ; |z| > 2$

8.2
$$X(z) = \frac{z^3 + 4z^2 - \frac{11}{6}}{z^3 + \frac{7}{6}z^2 - \frac{3}{2}z + \frac{1}{3}}$$

$$= \frac{(z + 3.8781)(z + 0.7512)(z - 0.6292)}{(z - \frac{1}{3})(z - \frac{1}{2})(z + 2)}$$

$$= \frac{\frac{1}{3}}{z - \frac{1}{3}} + \frac{\frac{1}{2}}{z - \frac{1}{2}} + \frac{2}{z + 2} + 1$$

$$= \frac{z}{z - \frac{1}{3}} + \frac{z}{z - \frac{1}{2}} - \frac{z}{z + 2}$$

(i) If ROC is $|z| < \frac{1}{3}$, all the poles are non causal.

$$\therefore Y(z) = -\left(\frac{1}{3}\right)^n u(-n) - \left(\frac{1}{2}\right)^n u(-n) + (-2)^n u(-n)$$

(ii) If ROC is $|z| > 3$, all the poles are causal

$$\therefore Y(z) = \left(\frac{1}{3}\right)^n u(n) + \left(\frac{1}{2}\right)^n u(n) - (-2)^n u(n)$$

8.3

(a)

$$Z[a^n \sin \Omega_0 n] = \frac{a z \sin \Omega_0}{z^2 - 2a z \cos \Omega_0 + a^2}$$

$$\therefore Z[n a^n \sin \Omega_0 n] = -z \frac{d}{dz} \left[\frac{a z \sin \Omega_0}{z^2 - 2a z \cos \Omega_0 + a^2} \right]$$

$$= \frac{a \sin \Omega_0 z^3 - a^3 \sin \Omega_0 z}{[z^2 - 2a z \cos \Omega_0 + a^2]^2}$$

$$(b) Z[n \cos \Omega_0 n] = -z \frac{d}{dz} \frac{z(z - \cos \Omega_0)}{z^2 - 2z \cos \Omega_0 + 1}$$

$$= \frac{z^3 \cos \Omega_0 - 2z^2 + z \cos \Omega_0}{[z^2 - 2z \cos \Omega_0 + 1]^2}$$

$$(c) X(z) = \frac{\frac{1}{2}z}{(z-1)^2} + \frac{1}{3}z^{-1} \frac{\frac{1}{3}z}{(z-\frac{1}{3})^2}$$

$$= \frac{\frac{1}{2}z}{(z-1)^2} + \frac{\frac{1}{9}}{(z-\frac{1}{3})^2}$$

$$(d) X(z) = z^{-2} + \frac{z}{(z-1)^2}$$

$$(e) Z[2 \sin \frac{2}{5} \pi n] = \frac{2z \sin \frac{2}{5} \pi}{z^2 - 2z \cos \frac{2}{5} \pi + 1}$$

$$Z \left[2 e^{-n} \sin \frac{2}{5} \pi n \right] = \frac{2(e^{-1}z) \sin \frac{2}{5} \pi}{(e^{-1}z)^2 - 2(e^{-1}z) \cos \frac{2\pi}{5} + 1}$$

8.4 (a) $x(nT) = nT \cos 1000 \pi nT$

$$X(z) = \frac{T [z^3 \cos(1000 \pi T) - 2z^2 + z \cos(1000 \pi T)]}{[z^2 - 2z \cos(1000 \pi T) + 1]^2}$$

from Prob 8.3 (c) with $\Omega_0 = 1000 \pi T$

(b) $x(nT) = nT \exp[-3(nT-1)] = Te^3 n e^{-3Tn}$

$$X(z) = \frac{e^3 T z \exp[-3T]}{[z - \exp[-3T]]^2} \text{ from entry 16 in Table 8-2}$$

8.5 (i)

$$\begin{array}{r} 1 - \frac{3}{4}z^{-1} + \frac{1}{8}z^{-2} \left| \begin{array}{l} 1 + \frac{5}{4}z^{-1} + \frac{13}{16}z^{-2} + \dots \\ 1 + \frac{1}{2}z^{-1} \\ 1 - \frac{3}{4}z^{-1} + \frac{1}{8}z^{-2} \\ \hline \frac{5}{4}z^{-1} - \frac{1}{8}z^{-2} \\ \frac{5}{4}z^{-1} - \frac{15}{16}z^{-2} + \frac{5}{32}z^{-3} \\ \hline \frac{13}{16}z^{-2} - \frac{5}{32}z^{-3} \\ \text{etc.} \end{array} \right. \end{array}$$

$$\therefore x(0) = 1, \quad x(1) = \frac{5}{4}, \quad x(2) = \frac{13}{16} \text{ etc.}$$

By Partial Fractions:

$$X(z) = \frac{1 + \frac{1}{2}z^{-1}}{(1 - \frac{1}{2}z^{-1})(1 - \frac{1}{4}z^{-1})} = \frac{4}{1 - \frac{1}{2}z^{-1}} + \frac{-3}{1 - \frac{1}{4}z^{-1}}$$

$$\therefore x(n) = 4 \left(\frac{1}{2}\right)^n u(n) - 3 \left(\frac{1}{4}\right)^n u(n)$$

$$\begin{array}{r} \text{(ii) } X(z) = z^2 + \frac{3}{4}z + \frac{1}{8} \\ \hline z^2 - \frac{5}{8}z + \frac{3}{32} \end{array}$$

By long division:

$$\begin{array}{r}
 1 + \frac{11}{8} z^{-1} + \frac{57}{64} z^{-2} + \dots \\
 \hline
 z^2 - \frac{5}{8} z + \frac{3}{32} \quad \left| \quad z^2 + \frac{3}{4} z + \frac{1}{8} \right. \\
 \hline
 z^2 - \frac{5}{8} z + \frac{3}{32} \\
 \hline
 \frac{11}{8} z + \frac{1}{32} \\
 \frac{11}{8} z - \frac{55}{64} + \frac{33}{256} z^{-1} \\
 \hline
 \frac{57}{64} - \frac{33}{256} z^{-1}
 \end{array}$$

$$\therefore x(0) = 1, \quad x(1) = \frac{11}{8}, \quad x(2) = \frac{57}{64} \dots$$

By Partial Fractions:

$$\begin{aligned}
 X(z) &= \frac{z(z + \frac{1}{2})(z + \frac{1}{4})}{z(z - \frac{3}{8})(z - \frac{1}{4})} \\
 &= \frac{4}{3} + \frac{35/3}{z - 3/8} + \frac{-12}{z^{-1}/4}
 \end{aligned}$$

$$x(n) = \frac{4}{3} \delta(n) + \left[\frac{35}{3} \left(\frac{3}{8}\right)^n - 12 \left(\frac{1}{4}\right)^n \right] u(n)$$

(iii)

$$\begin{array}{r}
 z^{-3} + \frac{3}{2} z^{-4} + \frac{3}{2} z^{-5} + \frac{5}{4} z^{-6} + \dots \\
 \hline
 z^3 - \frac{3}{2} z^2 + \frac{3}{4} z - \frac{1}{8} \quad \left| \quad 1 \right. \\
 \hline
 1 - \frac{3}{2} z^{-1} + \frac{3}{4} z^{-2} - \frac{1}{8} z^{-3} \\
 \hline
 \frac{3}{2} z^{-1} - \frac{3}{4} z^{-2} + \frac{1}{8} z^{-3} \\
 \hline
 \frac{3}{2} z^{-1} - \frac{9}{4} z^{-2} + \frac{9}{8} z^{-3} - \frac{3}{16} z^{-4} \\
 \hline
 \frac{3}{2} z^{-2} - z^{-3} + \frac{3}{16} z^{-4} \\
 \hline
 \frac{3}{2} z^{-2} - \frac{9}{4} z^{-3} + \frac{9}{8} z^{-4} - \frac{3}{16} z^{-5} \\
 \hline
 \frac{5}{4} z^{-3} - \frac{15}{16} z^{-4} + \frac{3}{16} z^{-5}
 \end{array}$$

$$x(0) = 0, x(1) = 0, x(2) = 0, x(3) = 1, x(4) = 3/2$$

$$x(5) = 3/2, x(6) = 5/4 \text{ etc.}$$

Entry 9 in Table 8.4.2 gives, with $m = 2$,

$$x(n) = \frac{(n-2)(n-1)}{2!} \left(\frac{1}{2}\right)^{n-3} u(n-3)$$

(iv)

$$\begin{array}{r} 1 - 2z^{-1} + 5z^{-2} - 14z^{-3} + \dots \\ z^2 + 4z + 3 \overline{) \quad} \\ \underline{z^2 + 2z} \\ -2z - 3 \\ \underline{-2z - 8} \\ 5 + 6z^{-1} \\ \underline{5 + 20z^{-1} + 15z^{-2}} \\ -14z^{-1} - 15z^{-2} \end{array}$$

$$\therefore x(0) = 1, x(1) = -2, x(2) = 5, x(3) = -14 \text{ etc.}$$

By partial fractions:

$$X(z) = \frac{z(z+2)}{(z+1)(z+3)} = \frac{\frac{1}{2}z}{z+1} + \frac{\frac{1}{2}z}{z+3}$$

$$x(n) = (-1)^n \frac{1}{2} u(n) + \frac{1}{2} (-3)^n u(n)$$

$$\begin{aligned} 8.6 (i) \log(1 - \tfrac{1}{3} z^{-1}) &= - \sum_{i=1}^{\infty} \frac{(\frac{1}{3} z^{-1})^i}{i} \\ &= - \left[\frac{1}{3} z^{-1} + \frac{1}{18} z^{-2} + \dots \right] \end{aligned}$$

$$\therefore x(0) = 0, x(1) = -\frac{1}{3} \text{ etc.}$$

$$\text{In general } x(n) = -\frac{1}{n} \left(\frac{1}{3}\right)^n \text{ for } n \neq 0$$

(ii) Let $y(n) = nx(n)$

$$\begin{aligned}\text{Then } Y(z) &= -z \frac{d}{dz} X(z) = -z \frac{1}{1 - \frac{1}{3}z^{-1}} \left(-\frac{1}{3}\right)(-z^{-2}) \\ &= -\frac{1}{3} \frac{1}{z - \frac{1}{3}}\end{aligned}$$

$$\therefore y(n) = -\frac{1}{3} \left(\frac{1}{3}\right)^{n-1} u(n-1)$$

$$= -\left(\frac{1}{3}\right)^n u(n-1)$$

$$\therefore x(n) = \frac{1}{n} y(n) = -\frac{1}{n} \left(\frac{1}{3}\right)^n u(n-1)$$

8.7 (i) $H(z) = \frac{z}{z - \frac{1}{2}}$, $X(z) = \frac{z(1 - z^{-11})}{z - 1}$

$$\therefore Y(z) = \frac{z^2(1 - z^{-11})}{(z - \frac{1}{2})(z - 1)} = (1 - z^{-11}) \left[\frac{-z}{z - \frac{1}{2}} + \frac{2z}{z - 1} \right]$$

$$y(n) = \left(-\frac{1}{2}\right)^n [u(n) - u(n-11)] + 2[u(n) - u(n-11)]$$

(ii) $H(z) = \frac{z}{z - \frac{1}{3}}$, $X(z) = \frac{z[1 - z^{-10}]}{z - 1} + \frac{z[1 - z^{-6}]}{z - 1}$

$$\therefore Y(z) = \frac{z^2}{(z - 1)(z - \frac{1}{3})} [(1 - z^{-10}) + (1 - z^{-6})]$$

$$= \left[\frac{\frac{3}{2}z}{z - 1} - \frac{\frac{1}{2}z}{z - \frac{1}{3}} \right] [(1 - z^{-10}) + (1 - z^{-6})]$$

$$= \left[\frac{3}{2} - \frac{1}{2} \left(\frac{1}{3}\right)^n \right] [2u(n) - u(n-6) - u(n-10)]$$

(iii) $H(z) = [1 - z^{-1} + 2z^{-2} - z^{-3} + z^{-4}]$

$$X(z) = [1 - 2z^{-2} + 3z^{-3}]$$

$$Y(z) = H(z)X(z) = 1 - z^{-1} + 4z^{-3} - 6z^{-4} + 8z^{-5} - 5z^{-6} + 3z^{-7}$$

$$\therefore y(n) = \{1, -1, 0, 4, -6, 8, -5, 3\}$$

$$8.8 \quad H(z) = \frac{z - 1/2}{(z + 1/2)(z + 1/4)} \quad X(z) = \frac{z}{z-1}$$

$$Y(z) = \frac{z(z - 1/2)}{(z-1)(z + 1/2)(z + 1/4)}$$

$$= \frac{\frac{4}{15} z}{z-1} - \frac{\frac{8}{3} z}{z + 1/2} + \frac{\frac{12}{5} z}{z + 1/4}$$

$$y(n) = \frac{4}{15} u(n) - \frac{8}{3} \left(-\frac{1}{2}\right)^n u(n) + \frac{12}{5} \left(-\frac{1}{4}\right)^n u(n)$$

8.9 (i) Transforming the difference equation gives

$$Y(z) - z^{-1}[Y(z) + z] + z^{-2}[Y(z) + z] = \frac{z}{z - \frac{1}{2}}$$

$$[1 - z^{-1} + z^{-2}]Y(z) = (1 - z^{-1}) + \frac{z}{z - \frac{1}{2}}$$

$$[z^2 - z + 1]Y(z) = z^2 - z + \frac{z^3}{z - \frac{1}{2}}$$

$$Y(z) = \frac{z(z-1)}{z^2 - z + 1} + \frac{z^3}{(z^2 - z + 1)(z - \frac{1}{2})}$$

$$= \frac{z(z-1)}{z^2 - z + 1} + \frac{\frac{2}{3}z(z+1)}{z^2 - z + 1} + \frac{\frac{1}{3}z}{z - \frac{1}{2}}$$

$$= \frac{\frac{5}{3}z^2 - \frac{1}{3}z}{z^2 - z + 1} + \frac{\frac{1}{3}z}{z - \frac{1}{2}}$$

$$= \frac{\frac{5}{3}z(z - \frac{1}{2})}{z^2 - z + 1} + \frac{1}{\sqrt{3}} \frac{\frac{\sqrt{3}}{2}z}{z^2 - z + 1} + \frac{1}{3} \frac{z}{z - \frac{1}{2}}$$

$$y(n) = \left[\frac{5}{3} \cos \frac{\pi}{3} n + \frac{1}{\sqrt{3}} \sin \frac{\pi}{3} n + \frac{1}{3} \left(\frac{1}{2}\right)^n \right] u(n)$$

(b) Transform the difference equation to get
 $Y(z) - \frac{1}{6} z^{-1} Y(z) - \frac{1}{6} z^{-2} Y(z) = \frac{z}{z - \frac{1}{3}}$

$$\therefore Y(z) = \frac{z^2}{z^2 - \frac{1}{6}z - \frac{1}{6}} \cdot \frac{z}{z - \frac{1}{3}} = \frac{\frac{9}{5}z}{z - \frac{1}{2}} + \frac{\frac{1}{5}z}{z + \frac{1}{3}} - \frac{z}{z - \frac{1}{3}}$$

$$y(n) = \left[\frac{9}{5} \left(\frac{1}{2}\right)^n + \frac{1}{5} \left(-\frac{1}{3}\right)^n - \left(\frac{1}{3}\right)^n \right] u(n)$$

8.10 (i) $[1 + z^{-1} + z^{-2}] Y(z) - 1 = \frac{1}{1 - z^{-1}}$

$$Y(z) = \frac{z^2}{z^2 + z + 1} + \frac{z^3}{(z-1)(z^2 + z + 1)}$$

$$= \frac{\frac{\sqrt{3}}{3} z^2 + \frac{1}{3} z}{z^2 + z + 1} + \frac{\frac{1}{3} z}{z-1}$$

$$= \frac{\frac{\sqrt{3}}{3} z^2 + \frac{1}{2} z}{z^2 + z + 1} - \frac{1}{\sqrt{3}} \frac{\frac{\sqrt{3}}{2} z}{z^2 + z + 1} + \frac{1}{3} \frac{z}{z-1}$$

$$\therefore y(n) = \frac{\sqrt{3}}{3} \cos \frac{2\pi}{3} n - \frac{1}{\sqrt{3}} \sin \frac{2\pi}{3} n + \frac{1}{3} u(n) \quad n \geq 0$$

(ii) $[1 - \frac{1}{4}z - \frac{1}{8}z^2] Y(z) - \frac{1}{4} - \frac{1}{8}z^{-1}$

$$= \frac{1}{1 - \frac{1}{2}z^{-1}} - \frac{\frac{1}{2}z^{-1}}{1 - \frac{1}{2}z^{-1}}$$

$$\therefore Y(z) = \frac{\frac{1}{4}z^2 + \frac{1}{8}z}{z^2 - \frac{1}{4}z - \frac{1}{8}} + \frac{z^2}{z^2 - \frac{1}{4}z - \frac{1}{8}}$$

$$= \frac{z}{z - \frac{1}{2}} + \frac{\frac{1}{4}z}{z + \frac{1}{4}}$$

$$y(n) = \left(\frac{1}{2}\right)^n + \left(-\frac{1}{4}\right)^n \quad n \geq 0$$

(iii) $[1 - z^{-1} + \frac{15}{64}z^{-2}] Y(z) + \frac{15}{64} = \frac{1}{1 - 2z^{-1}}$

$$Y(z) = -\frac{\frac{15}{64}z^2}{z^2 - z + \frac{15}{64}} + \frac{z^2}{(z-2)(z^2 - z + \frac{15}{64})}$$

$$= \frac{256/143 z}{z-2} + \frac{12771}{18304} \frac{z}{z - 3/8} + \frac{31525}{18304} \frac{z}{z - 5/8}$$

$$y(n) = \frac{256}{143} (2)^n + \frac{12771}{18304} \left(\frac{3}{8}\right)^n + \frac{31525}{18304} \left(\frac{5}{8}\right)^n$$

$$(iv) \left[z^2 + \frac{2}{3}z + \frac{1}{9} \right] Y(z) - z = \frac{z}{z - 1/2}$$

$$\begin{aligned} Y(z) &= \frac{z^2 + \frac{1}{2}z}{z^2 + \frac{2}{3}z + \frac{1}{9}} \\ &= \frac{36/25 z}{z - 1/2} + \frac{-36/25 z}{(z - 1/3)} + \frac{-1/5 z}{(z - 1/3)^2} \end{aligned}$$

$$y(n) = \frac{36}{25} \left(\frac{1}{2}\right)^n - \frac{36}{25} \left(\frac{1}{3}\right)^n - \frac{3}{5} n \left(\frac{1}{3}\right)^n \quad n \geq 0$$

$$(v) Y(z) = [1 - 3z^{-1} + 2z^{-2} - z^{-3}] \frac{z}{z-1}$$

$$y(n) = u(n) - 3u(n-1) + 2u(n-2) - u(n-3)$$

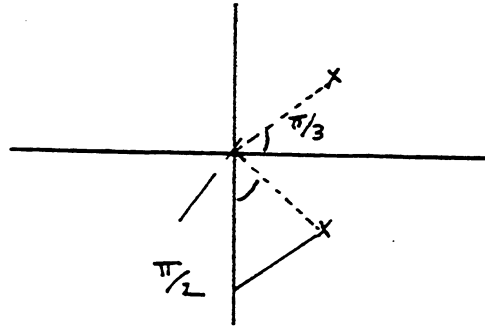
$$= \begin{cases} 1 & n=0 \\ -2 & n=1 \\ 0 & n=2 \\ -1 & n \geq 3 \end{cases}$$

$$\begin{aligned} 8.11(i) \quad H(z) &= \frac{1}{1 - \frac{1}{2}z^{-1} + \frac{1}{4}z^{-2}} = \frac{z^2}{z^2 - \frac{1}{2}z + \frac{1}{4}} \\ &= \frac{z(z - \frac{1}{4})}{z^2 - \frac{1}{2}z + \frac{1}{4}} + \frac{\frac{1}{4}z}{z^2 - \frac{1}{2}z + \frac{1}{4}} \end{aligned}$$

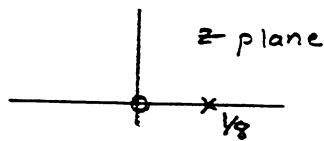
Use entries 10 and 11 in Table 8.4.2 with

$$a = \frac{1}{2}, \quad \Omega_0 = \frac{\pi}{3}$$

$$\therefore h(n) = \left[\left(\frac{1}{2}\right)^n \cos \frac{n\pi}{3} + \frac{1}{4} \left(\frac{1}{2}\right)^n \sin \frac{n\pi}{3} \right] u(n)$$



$$(ii) \quad H(z) = \frac{1 - \frac{1}{4}z^{-1}}{1 - \frac{3}{8}z^{-1} + \frac{1}{32}z^{-2}} = \frac{3}{z - \frac{1}{8}}$$



$$h(n) = \left(\frac{1}{8}\right)^n u(n)$$

$$8.12 \quad (a) \quad x(z) = \frac{z}{z-1} + \frac{z}{z+\frac{1}{2}} = \frac{2z(z-\frac{1}{4})}{(z-1)(z+\frac{1}{2})}$$

$$Y(z) = \frac{6z}{z+\frac{1}{4}} - \frac{6z}{z+\frac{1}{3}} = \frac{\frac{1}{2}z}{(z+\frac{1}{4})(z+\frac{1}{3})}$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1}{4} \frac{(z-1)(z+\frac{1}{2})}{(z+\frac{1}{4})(z-\frac{1}{4})(z+\frac{1}{3})}$$

$$= \frac{\frac{1}{4}z^{-1} - \frac{1}{8}z^{-2} - \frac{1}{8}z^{-3}}{1 + \frac{1}{3}z^{-1} - \frac{1}{16}z^{-2} - \frac{1}{48}z^{-3}}$$

(b) The difference equation for the system is

$$y(n) + \frac{1}{3}y(n-1) - \frac{1}{16}y(n-2) - \frac{1}{48}y(n-3)$$

$$= \frac{1}{4}x(n-1) - \frac{1}{8}x(n-2) - \frac{1}{8}x(n-3)$$

$$8.13 \quad H_1(z) = \frac{z}{(z-1)^2} - \frac{z}{z-1} = \frac{2z - z^2}{(z-1)^2}$$

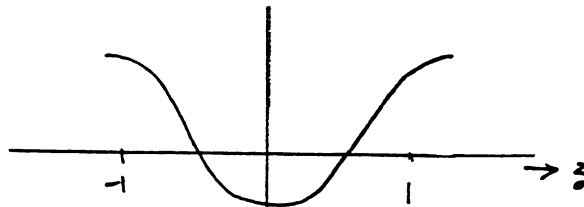
$$H_2(z) = 1 + \frac{z}{(z-1)^2} + z^{-2} = \frac{z^4 - z^3 + 2z^2 - 2z + 1}{z^2(z-1)^2}$$

$$H_3(z) = \frac{z}{z - 1/2}$$

$$\begin{aligned} H(z) &= H_3(z) [H_2(z) - H_1(z)] \\ &= \frac{z}{z - 1/2} \left[\frac{2z^4 - z^3 + 2z^2 - 2z + 1}{z^2(z-1)^2} \right] \\ &= \frac{2z^3 - z^2 + z - 1}{z(z - 1/2)(z-1)} \end{aligned}$$

8.14 (a) Assuming a_1 and a_2 are real, $F(z)$ will have (i) both roots real or (ii) roots form a complex conjugate pair.

In the first case, for roots within the unit circle, the sketch of $F(z)$ vs z will be



$$\therefore F(-1) > 0, F(1) > 0$$

Now with both roots complex conjugate,

$$\begin{aligned} F(z) &= (z - re^{j\theta})(z - re^{-j\theta}) \\ &= z^2 - 2rz \cos \theta + r^2 \end{aligned}$$

For roots inside unit circle $|r| < 1$

$$\therefore |F(0)| = r^2 < 1$$

$$(b) F(z) = z^2 + (.8K - 1.3)z + .04$$

For stability

$$F(-1) = 1 - (.8K - 1.3) + .04 = 2.34 - .8K > 0$$

$$\text{or } K < 2.925$$

$$F(1) = 1 + (.8K - 1.3) + .04 > 0$$

$$\text{or } K > .325$$

Since $|F(0)| < 1$ for all K , system is stable

$$\text{if } .325 < K < 2.925$$

$$8.15 \quad \text{Let } F(z) = z^2 + (K - 1 - \alpha)z + (1 - K)\alpha$$

$$|F(0)| = |1 + K - 1 - \alpha + (1 - K)\alpha| = |K(1 - \alpha)|$$

$$F(-1) = 1 - (K - 1 - \alpha) + (1 - K)\alpha = 2(1 + \alpha) = K(1 + \alpha)$$

$$F(1) = 1 + (K - 1 - \alpha) + (1 - K)\alpha = K(1 - \alpha)$$

$$|F(0)| < 1 \quad \text{gives} \quad -1 < K(1 - \alpha) < 1$$

$$F(1) > 0 \quad \text{gives} \quad K(1 - \alpha) > 0$$

$$\therefore 0 < K(1 - \alpha) < 1 \quad (i)$$

$$F(-1) > 0 \quad \text{gives} \quad 2(1 + \alpha) > K(1 + \alpha)$$

$$\text{If } \alpha > -1, \text{ we get } K < 2 \quad (ii)$$

$$\text{If } \alpha < -1, \text{ we get } K > 2 \quad (iii)$$

$$\text{If } K > 2, \text{ from (i), } 1 - \alpha < \frac{1}{K} < \frac{1}{2}$$

$$\therefore \alpha > \frac{1}{2} \text{ which contradicts (iii)}$$

$$\text{If } 0 < K < 2, \text{ from (i) we have } 1 - \alpha > 0$$

$$\therefore \alpha < 1$$

$$\text{Also } K < \frac{1}{1 - \alpha} \quad \therefore 2 < \frac{1}{1 - \alpha} \text{ or } 1 - \alpha < \frac{1}{2}$$

$$\therefore \alpha > \frac{1}{2}$$

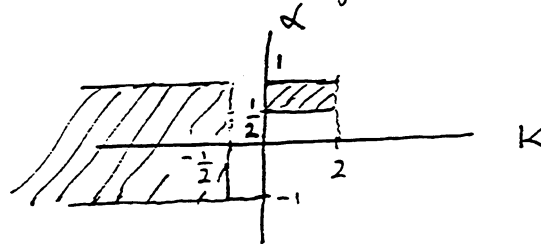
$$\therefore \text{For } 0 < K < 2, \quad \frac{1}{2} < \alpha < 1$$

For $K < 0$, from (i) $1 - \alpha > 0 \therefore \alpha < 1$

From (ii), $\alpha > -1 \therefore -1 < \alpha < 1$

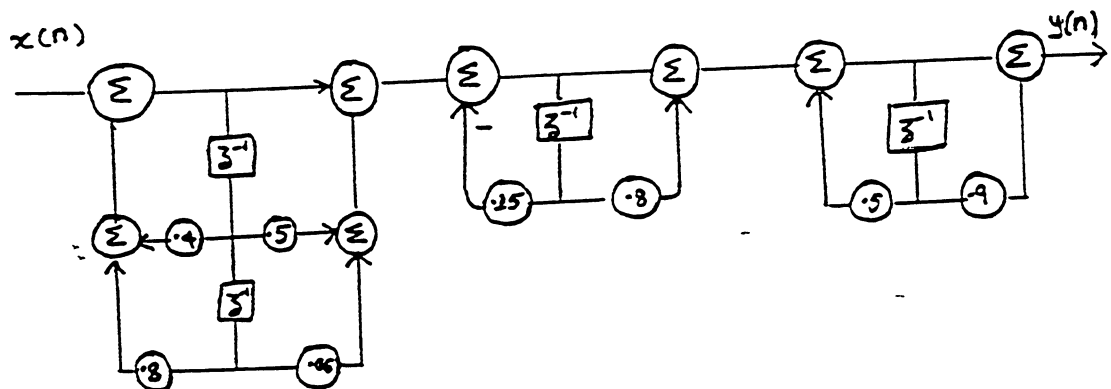
Also from (i) $1 - \alpha > \frac{1}{K} \therefore -\infty < K < -\frac{1}{2}$

$\therefore$ For stability, if $0 < K < 2$, $\frac{1}{2} < \alpha < 1$
 if $-\infty < K < -\frac{1}{2}$, $-1 < \alpha < 1$

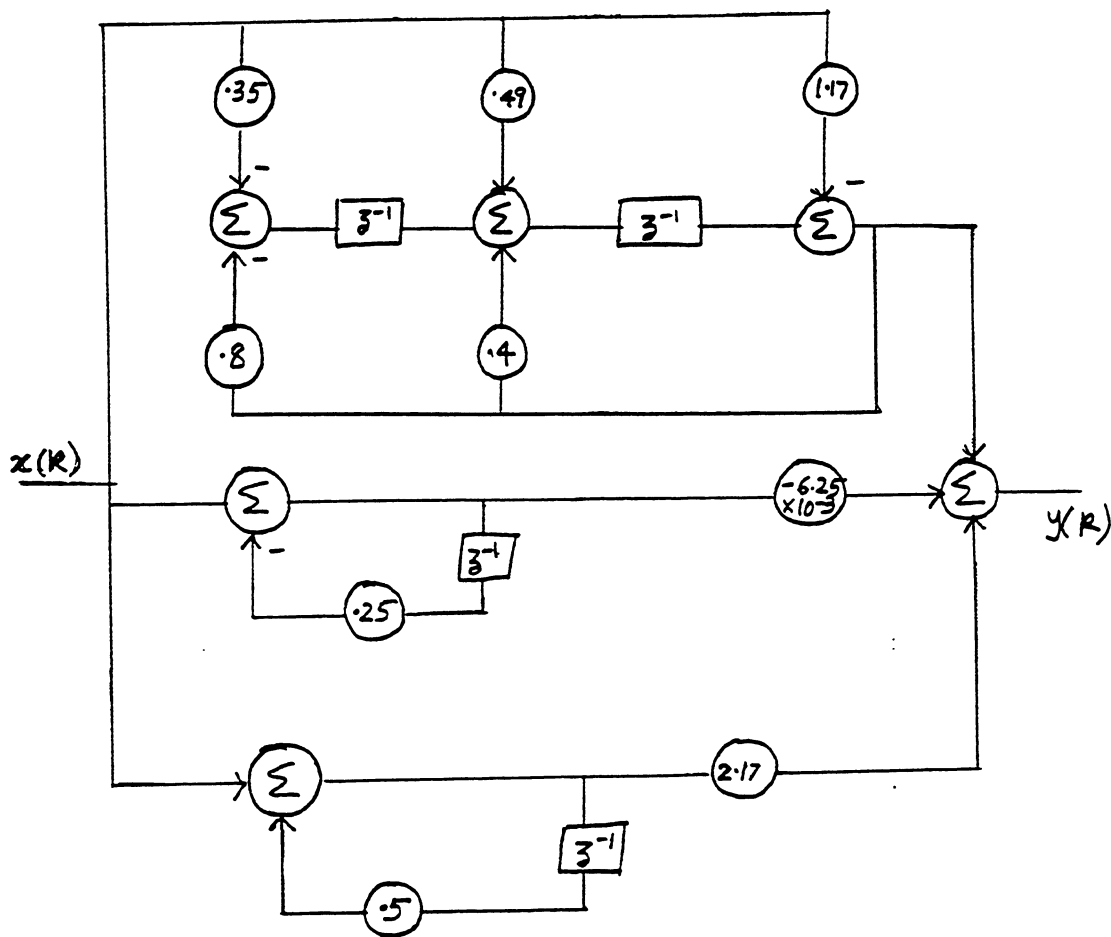


/// - stable region

$$\begin{aligned}
 8.16 \quad (i) \quad H(z) &= \frac{(1 + .3z^{-1})(1 + .2z^{-1})(1 + .8z^{-1})(1 + .9z^{-1})}{(1 + .4z^{-1} + 0.8z^{-2})(1 + 0.25z^{-1})(1 - .5z^{-1})} \\
 &= \frac{1 + .5z^{-1} + .66z^{-2}}{1 + .4z^{-1} + .8z^{-2}} \cdot \frac{1 + .8z^{-1}}{1 + .25z^{-1}} \cdot \frac{1 + .9z^{-1}}{1 - .5z^{-1}}
 \end{aligned}$$



$$(ii) \quad H(z) = \frac{-1.17 + .49z^{-1} - .35z^{-2}}{1 + .4z^{-1} + .8z^{-2}} + \frac{-6.25 \times 10^{-3}}{1 + .25z^{-1}} + \frac{2.17}{1 - .5z^{-1}}$$



8.17 Use 2nd Canonical form in all cases

$$(i) \quad A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{1}{24} & -\frac{9}{24} & \frac{13}{12} \end{bmatrix} \quad b = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad c = [1 \ 0 \ 0] \quad d = 0$$

$$(a) \phi(k) = Z^{-1} [Z(ZI - A)^{-1}]$$

$$= Z^{-1} \begin{bmatrix} \frac{28}{3} \frac{z}{z+\frac{1}{2}} - \frac{61}{7} \frac{z}{z+\frac{1}{3}} + \frac{8}{21} \frac{z}{z-\frac{1}{4}} & \frac{-7z}{z+\frac{1}{2}} + \frac{46z}{z+\frac{1}{3}} + \frac{35z}{z-\frac{1}{4}} & \frac{-4z}{z+\frac{1}{2}} + \frac{24z}{z+\frac{1}{3}} + \frac{12z}{z-\frac{1}{4}} \\ \frac{-3z}{z+\frac{1}{2}} + \frac{3z}{z+\frac{1}{3}} + \frac{3z}{z-\frac{1}{4}} & \frac{-7z}{z+\frac{1}{2}} + \frac{-6z}{z+\frac{1}{3}} + \frac{5z}{z-\frac{1}{4}} & \frac{2z}{z+\frac{1}{2}} + \frac{-8z}{z+\frac{1}{3}} + \frac{1}{z-\frac{1}{4}} \\ \frac{z}{z+\frac{1}{2}} + \frac{-3z}{z+\frac{1}{3}} + \frac{1}{z-\frac{1}{4}} & \frac{-11z}{z+\frac{1}{2}} + \frac{4z}{z+\frac{1}{3}} + \frac{-5z}{z-\frac{1}{4}} & \frac{z}{z+\frac{1}{2}} + \frac{8z}{z+\frac{1}{3}} + \frac{1}{z-\frac{1}{4}} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{28}{3} \left(\frac{1}{2}\right)^k - \frac{61}{7} \left(\frac{1}{3}\right)^k + \frac{8}{21} \left(\frac{1}{4}\right)^k & \frac{-7}{3} \left(\frac{1}{2}\right)^k + \frac{46}{21} \left(\frac{1}{3}\right)^k + \frac{35}{84} \left(\frac{1}{4}\right)^k & \frac{-4}{3} \left(\frac{1}{2}\right)^k + \frac{24}{7} \left(\frac{1}{3}\right)^k + \frac{12}{27} \left(\frac{1}{4}\right)^k \\ \frac{-1}{6} \left(\frac{1}{2}\right)^k + \frac{1}{7} \left(\frac{1}{3}\right)^k + \frac{1}{42} \left(\frac{1}{4}\right)^k & \frac{-7}{6} \left(\frac{1}{2}\right)^k - \frac{6}{7} \left(\frac{1}{3}\right)^k + \frac{5}{42} \left(\frac{1}{4}\right)^k & \frac{-2}{3} \left(\frac{1}{2}\right)^k - \frac{8}{7} \left(\frac{1}{3}\right)^k + \frac{1}{7} \left(\frac{1}{4}\right)^k \\ \frac{1}{12} \left(\frac{1}{2}\right)^k - \frac{1}{21} \left(\frac{1}{3}\right)^k + \frac{1}{168} \left(\frac{1}{4}\right)^k & \frac{-11}{12} \left(\frac{1}{2}\right)^k + \frac{4}{7} \left(\frac{1}{3}\right)^k - \frac{5}{168} \left(\frac{1}{4}\right)^k & \left(\frac{1}{2}\right)^k + \frac{8}{21} \left(\frac{1}{3}\right)^k + \frac{1}{28} \left(\frac{1}{4}\right)^k \end{bmatrix}$$

$$(b) \text{ With } X(z) = \frac{z}{z-1}, \quad y(0) = 0$$

$$Y(z) = [ZI - A]^{-1} b X(z) = \frac{1}{(z+\frac{1}{2})(z+\frac{1}{3})(z-\frac{1}{4})} \begin{bmatrix} 1 \\ 3 \\ z^2 \end{bmatrix} \frac{z}{z-1}$$

$$= \begin{bmatrix} \frac{\frac{2}{3}z}{z-1} + \frac{\frac{-11}{3}z}{z+\frac{1}{2}} + \frac{\frac{54}{7}z}{z+\frac{1}{3}} + \frac{\frac{-64}{21}z}{z-\frac{1}{4}} \\ \frac{\frac{2}{3}z}{z-1} + \frac{\frac{8}{3}z}{z+\frac{1}{2}} - \frac{\frac{18}{7}z}{z+\frac{1}{3}} - \frac{\frac{16}{21}z}{z-\frac{1}{4}} \\ \frac{\frac{2}{3}z}{z-1} - \frac{\frac{4}{3}z}{z+\frac{1}{2}} + \frac{\frac{6}{7}z}{z+\frac{1}{3}} - \frac{\frac{4}{21}z}{z-\frac{1}{4}} \end{bmatrix}$$

$$\therefore \underline{v}(k) = \begin{bmatrix} \frac{2}{3} - \frac{16}{3} \left(-\frac{1}{2}\right)^k + \frac{54}{7} \left(-\frac{1}{3}\right)^k - \frac{64}{21} \left(\frac{1}{4}\right)^k \\ \frac{2}{3} + \frac{8}{3} \left(-\frac{1}{2}\right)^k - \frac{18}{7} \left(-\frac{1}{3}\right)^k - \frac{16}{21} \left(\frac{1}{4}\right)^k \\ \frac{2}{3} - \frac{4}{3} \left(-\frac{1}{2}\right)^k + \frac{6}{7} \left(-\frac{1}{3}\right)^k - \frac{4}{21} \left(\frac{1}{4}\right)^k \end{bmatrix}$$

$$y(k) = v_1(k)$$

$$\begin{aligned} \text{cc) } H(z) &= C [zI - A]^{-1} b \\ &= [1 \ 0 \ 0] [zI - A]^{-1} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \end{aligned}$$

$$= \frac{1}{(z + \frac{1}{2})(z + \frac{1}{3})(z - \frac{1}{4})}$$

$$\begin{aligned} \text{(ii) } A &= \begin{bmatrix} 0 & 1 \\ -1 & -0.707 \end{bmatrix} \quad b = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad C = \begin{bmatrix} -0.207 & -0.75 \end{bmatrix} \\ d &= 1 \end{aligned}$$

$$\begin{aligned} \text{(a)} \\ \phi(k) &= z^{-1} \begin{bmatrix} \frac{3(z + 0.707)}{z^2 + 0.707z + 1} & \frac{3}{z^2 + 0.707z + 1} \\ \frac{-3}{z^2 + 0.707z + 1} & \frac{z^2}{z^2 + 0.707z + 1} \end{bmatrix} \end{aligned}$$

$$= \begin{bmatrix} \cos(1.932k) + 0.378 \sin(1.932k) & 1.069 \sin(1.932k) \\ -1.069 \sin(1.932k) & \cos(1.932k) - 0.378 \sin(1.932k) \end{bmatrix}$$

$$(b) \quad V(z) = \begin{bmatrix} \frac{z^2}{(z-1)(z^2 + .707z + 1)} \\ \frac{z^3}{(z-1)(z^2 + .707z + 1)} \end{bmatrix} = \begin{bmatrix} \frac{Az+B}{z^2 + .707z + 1} + \frac{C}{z-1} \\ \frac{-Bz+C}{z^2 + .707z + 1} + \frac{C}{z-1} \end{bmatrix}$$

where $A = -.3693$, $B = -.6306$, $C = .3693$

$$\therefore V(k) = \begin{bmatrix} -.3693 \cos(1.932k) - .5238 \sin(1.932k) + .3693 \\ -.6306 \cos(1.932k) + .6304 \sin(1.932k) + .3693 \end{bmatrix}$$

and $y(k) = -.207 V_1(k) - 0.75 V_2(k) + 1$

$$(c) \quad H(z) = \mathcal{L} \{ (zI - A)^{-1} b \} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{\sqrt{2}} & -\frac{3}{4} \end{bmatrix} \begin{bmatrix} \frac{1}{z^2 + .707z + 1} + 1 \\ \frac{z}{z^2 + .707z + 1} \end{bmatrix}$$

$$= \frac{z^2 + \frac{1}{2}z + \frac{1}{4}}{z^2 + .707z + 1}$$

(iii) (a) $A = \begin{bmatrix} 0 & 1 \\ -2 & 3 \end{bmatrix}$ $b = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ $c = [1 \ 0]$ $d = 0$

$$\Phi(k) = z^{-1} \begin{bmatrix} \frac{2z}{z-1} - \frac{z}{z-2} & \frac{-z}{z-1} + \frac{z}{z-2} \\ \frac{+2z}{z-1} - \frac{2z}{z-2} & \frac{-z}{z-1} + \frac{2z}{z-2} \end{bmatrix}$$

$$= \begin{bmatrix} 2 - (2)^k & -1 + (2)^k \\ 2 - 2(2)^k & -1 + 2(2)^k \end{bmatrix}$$

$$(b) \quad V(z) = \begin{bmatrix} \frac{-3}{(z-1)^2} + \frac{3}{(z-1)(z-2)} \\ \frac{-3}{(z-1)^2} + \frac{23}{(z-1)(z-2)} \end{bmatrix}$$

$$\therefore V(k) = \begin{bmatrix} -k-1 + (2)^k \\ -k-2 + 2(2)^k \end{bmatrix}$$

$$y(k) = -k-1 + (2)^k$$

$$8.18 (i) \quad A = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} \quad b = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad c = [1 \quad 0]$$

$$\Phi(z) = Z^{-1} \begin{bmatrix} \frac{z+1}{z^2+z+1} & \frac{1}{z^2+z+1} \\ \frac{-1}{z^2+z+1} & \frac{z}{z^2+z+1} \end{bmatrix}$$

$$= \begin{bmatrix} \left[\cos \frac{2\pi}{3}(k-1) + \frac{1}{\sqrt{3}} \sin \frac{2\pi}{3}(k-1) \right] u(k-1) & \frac{2}{\sqrt{3}} \sin \frac{2\pi}{3}(k-1) u(k-1) \\ -\frac{2}{\sqrt{3}} \sin \left(\frac{2\pi}{3}(k-1) \right) u(k-1) & \frac{2}{\sqrt{3}} \sin \frac{2\pi}{3} k u(k) \end{bmatrix}$$

$$V(z) = \begin{bmatrix} \frac{3}{(z^2+z+1)(z-1)} \\ \frac{3^2}{(z^2+z+1)(z-1)} \end{bmatrix} = \begin{bmatrix} \frac{\frac{1}{3}z}{z-1} + \frac{-\frac{1}{3}z^2 - \frac{2}{3}z}{z^2+z+1} \\ \frac{\frac{1}{3}z}{z-1} - \frac{\frac{1}{3}z^2-1}{z^2+z+1} \end{bmatrix}$$

$$V(k) = \begin{bmatrix} \frac{1}{3} - \frac{1}{3} \cos \frac{2\pi}{3} k - \sqrt{3} \sin \frac{2\pi}{3} k \\ \frac{1}{3} - \frac{1}{3} \cos \frac{2\pi}{3} k + \frac{1}{\sqrt{3}} \sin \frac{2\pi}{3} k \end{bmatrix} \quad y(k) = v_1(k)$$

$$(ii) \quad A = \begin{bmatrix} 0 & 1 \\ 1/8 & 1/4 \end{bmatrix} \quad b = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad c = \begin{bmatrix} -1/2 & 1 \end{bmatrix}$$

$$\phi(z) = z^{-1} \begin{bmatrix} \frac{3(3 - \frac{1}{4})}{(3 - \frac{1}{2})(3 + \frac{1}{4})} & \frac{3}{(3 - \frac{1}{2})(3 + \frac{1}{4})} \\ \frac{1/8}{(3 - \frac{1}{2})(3 + \frac{1}{4})} & \frac{3^2}{(3 - \frac{1}{2})(3 + \frac{1}{4})} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{2}{3} \left(\frac{1}{2}\right)^k + \frac{1}{3} \left(-\frac{1}{4}\right)^k & \frac{4}{3} \left(\frac{1}{2}\right)^k - \frac{4}{3} \left(-\frac{1}{4}\right)^k \\ \frac{1}{6} \left(\frac{1}{2}\right)^k - \frac{4}{6} \left(-\frac{1}{4}\right)^k & \frac{2}{3} \left(\frac{1}{2}\right)^k + \frac{1}{3} \left(-\frac{1}{4}\right)^k \end{bmatrix}$$

$$V(z) = z^{-1} \begin{bmatrix} \frac{3^2}{(3 - \frac{1}{2})(3 + \frac{1}{4})(3 - 1)} \\ \frac{3^3}{(3 - \frac{1}{2})(3 + \frac{1}{4})(3 - 1)} \end{bmatrix} = \begin{bmatrix} \frac{-\frac{4}{3} 3}{3 - \frac{1}{2}} & \frac{\frac{4}{15} 3}{3 + \frac{1}{4}} & -\frac{\frac{8}{5} 3}{3 - 1} \\ \frac{-\frac{2}{3} 3}{3 - \frac{1}{2}} & \frac{\frac{1}{15} 3}{3 + \frac{1}{4}} & \frac{\frac{24}{15} 3}{3 - 1} \end{bmatrix}$$

$$V(k) = \begin{bmatrix} -\frac{4}{3} \left(\frac{1}{2}\right)^k - \frac{4}{15} \left(-\frac{1}{4}\right)^k - 1 \\ -\frac{2}{3} \left(\frac{1}{2}\right)^k + \frac{1}{15} \left(-\frac{1}{4}\right)^k + \frac{24}{15} \end{bmatrix}$$

$$y(k) = -\frac{1}{2} V_1(k) + V_2(k) = \frac{1}{5} \left(-\frac{1}{4}\right)^k + \frac{21}{10}$$

$$(iii) \quad A = \begin{bmatrix} 0 & 1 \\ -15/64 & 1 \end{bmatrix} \quad b = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad c = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

$$\phi(z) = z^{-1} \begin{bmatrix} \frac{3(3 - 1)}{(3 - \frac{5}{8})(3 - \frac{3}{8})} & \frac{3}{(3 - \frac{5}{8})(3 - \frac{3}{8})} \\ \frac{-15/64 \cdot 3}{(3 - \frac{5}{8})(3 - \frac{3}{8})} & \frac{3^2}{(3 - \frac{5}{8})(3 - \frac{3}{8})} \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{3}{2} \left(\frac{5}{8}\right)^k + \frac{5}{2} \left(\frac{3}{8}\right)^k & 4 \left(\frac{5}{8}\right)^k - 4 \left(\frac{3}{8}\right)^k \\ -\frac{15}{16} \left(\frac{5}{8}\right)^k + \frac{15}{16} \left(\frac{3}{8}\right)^k & \frac{5}{2} \left(\frac{5}{8}\right)^k - \frac{3}{2} \left(\frac{3}{8}\right)^k \end{bmatrix}$$

$$V(z) = \begin{bmatrix} \frac{3}{(z - \frac{5}{8})(z - \frac{3}{8})(z-1)} \\ \frac{z^2}{(z - \frac{5}{8})(z - \frac{3}{8})(z-1)} \end{bmatrix} = \begin{bmatrix} \frac{-\frac{32}{3} z}{z - \frac{5}{8}} + \frac{\frac{32}{5} z}{z - \frac{3}{8}} + \frac{\frac{64}{15} z}{z-1} \\ \frac{-\frac{20}{3} z}{z - \frac{5}{8}} + \frac{\frac{12}{5} z}{z - \frac{3}{8}} + \frac{\frac{64}{15} z}{z-1} \end{bmatrix}$$

$$V(k) = \begin{bmatrix} -\frac{32}{3} \left(\frac{5}{8}\right)^k + \frac{32}{5} \left(\frac{3}{8}\right)^k + \frac{64}{15} \\ -\frac{20}{3} \left(\frac{5}{8}\right)^k + \frac{12}{5} \left(\frac{3}{8}\right)^k + \frac{64}{15} \end{bmatrix}$$

$$y(k) = v_1(k)$$

$$(iv) \quad A = \begin{bmatrix} 0 & 1 \\ -\frac{1}{9} & -\frac{2}{3} \end{bmatrix} \quad b = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad c = [1 \quad 0]$$

$$\phi(n) = z^{-1} \begin{bmatrix} \frac{z(z + \frac{2}{3})}{(z + \frac{1}{3})^2} & \frac{-z}{(z + \frac{1}{3})^2} \\ \frac{\frac{1}{9} z}{(z + \frac{1}{3})^2} & \frac{z^2}{(z + \frac{1}{3})^2} \end{bmatrix}$$

$$= \begin{bmatrix} (1-k) \left(-\frac{1}{3}\right)^k & 3k \left(-\frac{1}{3}\right)^k \\ -\frac{1}{3} k \left(-\frac{1}{3}\right)^k & (k+1) \left(-\frac{1}{3}\right)^k \end{bmatrix}$$

$$V(z) = \begin{bmatrix} \frac{z}{(z-1)(z+\frac{1}{3})^2} \\ \frac{z^2}{(z-1)(z+\frac{1}{3})^2} \end{bmatrix} \cdot \begin{bmatrix} \frac{\frac{9}{16}z}{z-1} - \frac{\frac{1}{16}z^2 + \frac{15}{16}z}{(z+\frac{1}{3})^2} \\ \frac{\frac{9}{16}z}{z-1} - \frac{\frac{9}{16}z^2 + \frac{1}{16}z}{(z+\frac{1}{3})^2} \end{bmatrix}$$

$$V(n) = \begin{bmatrix} \frac{9}{16} + \frac{9}{4}k\left(-\frac{1}{3}\right)^k - \frac{9}{16}\left(-\frac{1}{3}\right)^k \\ \frac{9}{16} - \frac{3}{8}k\left(-\frac{1}{3}\right)^k - \frac{9}{16}\left(-\frac{1}{3}\right)^k \end{bmatrix}$$

$$Y(k) = V_1(k)$$

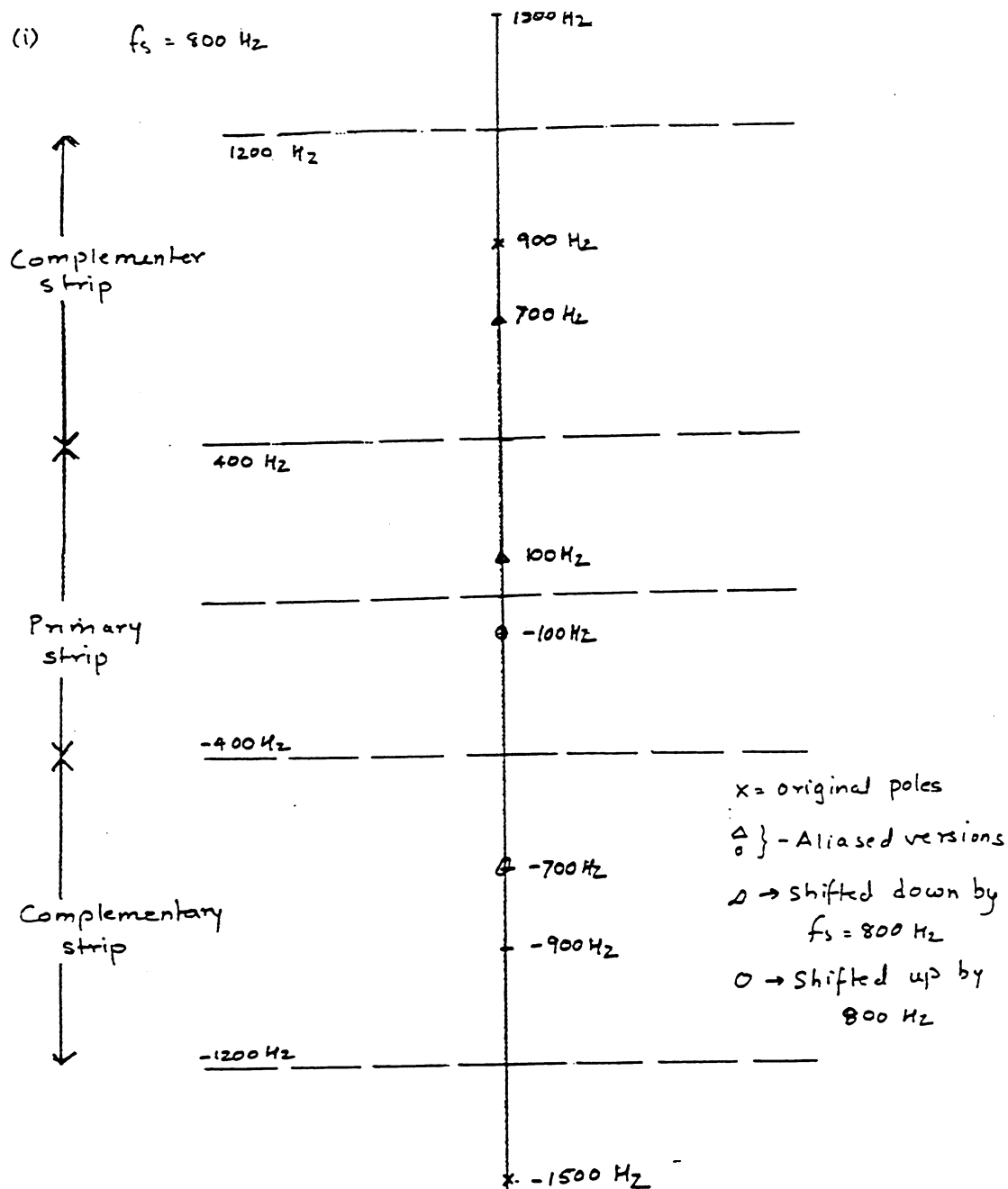
(v) Not a proper transfer function

$$8.19 (a) f_s \geq 6 \text{ kHz}, T \leq \frac{1}{6000} \text{ Secs}$$

(b) Range of frequencies mapped into unit circle

$$\text{is } \left[-\frac{f_s}{2}, \frac{f_s}{2}\right] = [-3 \text{ kHz}, 3 \text{ kHz}]$$

8.20 (i) $f_s = 800 \text{ Hz}$

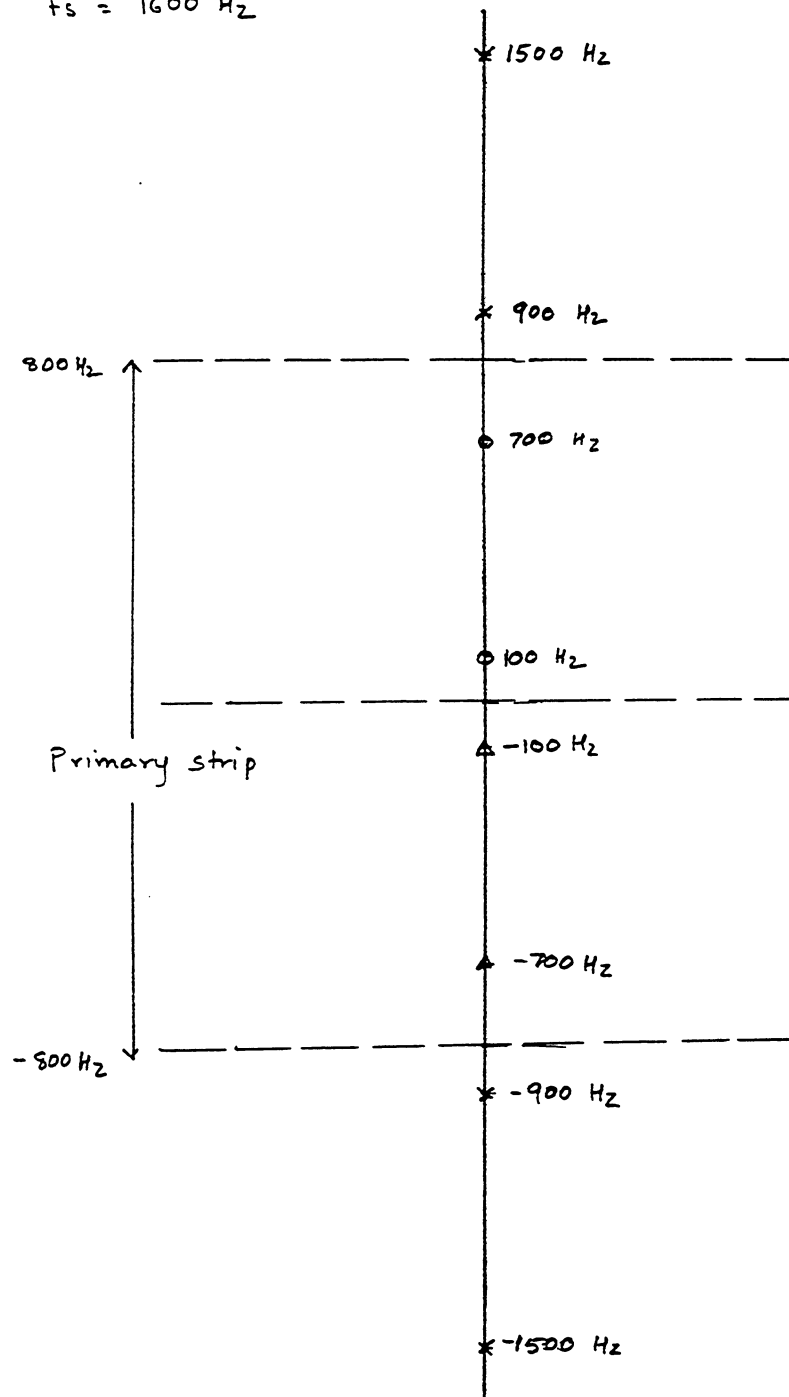


If we use a 1-p. filter which passes frequencies only up to 1500 Hz , output will have components at

100, 700, 900 and 1500 Hz

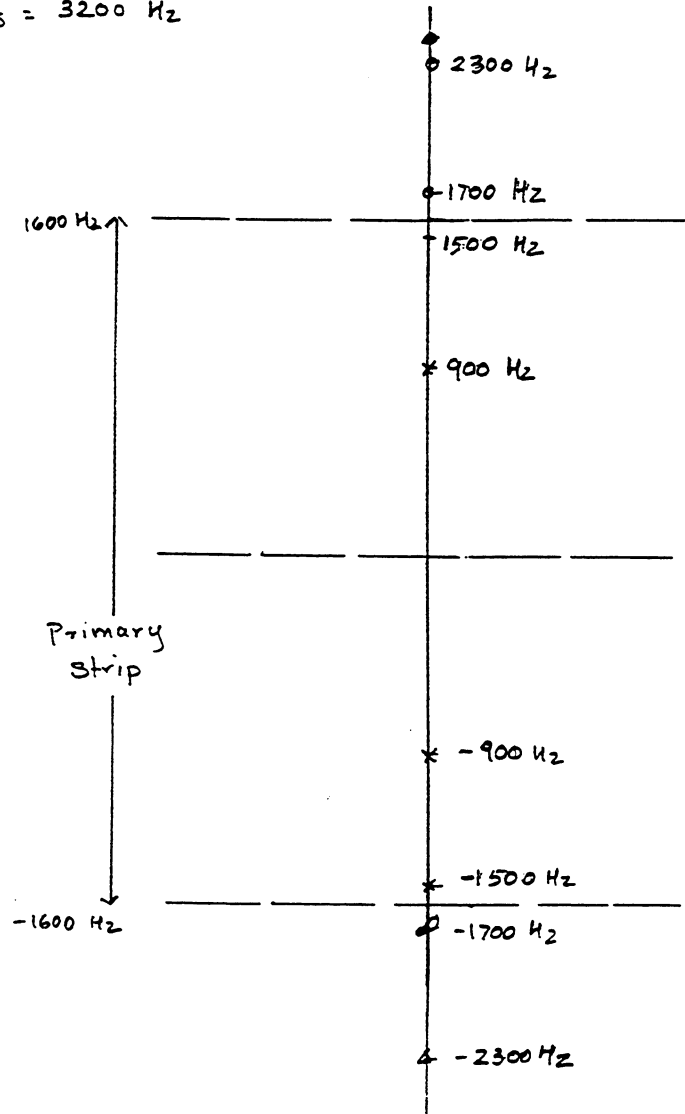
Distortion Desired components

(ii) $f_s = 1600 \text{ Hz}$



L.P Filter with Cut off 1500 Hz will give an output with components at 100 Hz , 700 Hz , 900 Hz , 1500 Hz

(iii) $f_s = 3200 \text{ Hz}$

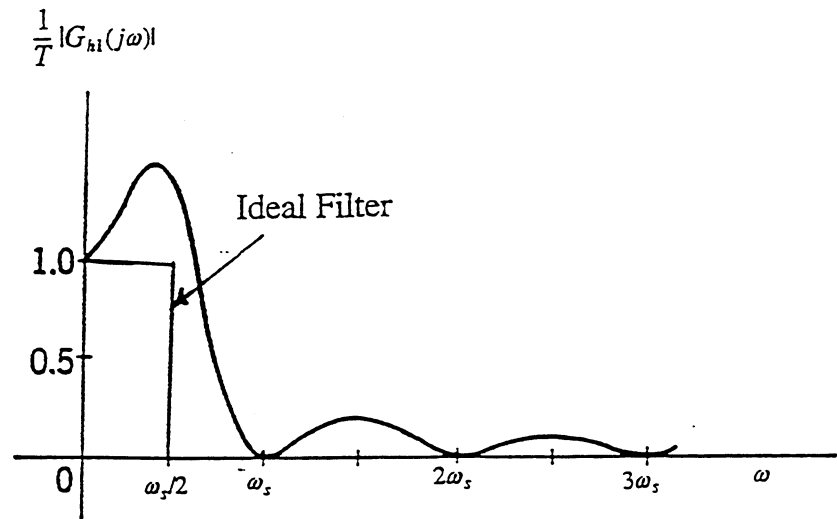


LPF at 1500 Hz passes only 900 Hz & 1500 Hz components - No aliasing.

8.21 To find the impulse response, let $x_a(0) = 1$,
 all other $x(nT) = 0$
 $\therefore$ For $0 \leq t \leq T$, $h(t) = 1 + \frac{t}{T}$

For $T \leq t < 2T$, $h(t) = \frac{t-T}{T} (-1) = 1 - \frac{t}{T}$

$$\begin{aligned} H(s) &= \int_0^T \left(1 + \frac{t}{T}\right) e^{-st} dt + \int_T^{2T} \left(1 - \frac{t}{T}\right) e^{-st} dt \\ &= \int_0^{2T} e^{-st} dt + \frac{1}{T} \int_0^T t e^{-st} dt - \frac{1}{T} \int_T^{2T} t e^{-st} dt \\ &= \frac{[1 - e^{-Ts}]^2 (1 + Ts)}{Ts^2} = \frac{1 + Ts}{T} \left[\frac{1 - e^{-Ts}}{s} \right]^2 \\ &= G_{hl}(s) \end{aligned}$$



8.22 (a) For the derivative operator $y(t) = \frac{d}{dt} x(t)$, we have $H(s) = s$

For the backward difference approximation,

$$Y(z) = T X(z) + z^{-1} Y(z)$$

$$\therefore H(z) = \frac{Y(z)}{X(z)} = \frac{1 - z^{-1}}{T} \quad \therefore s = \frac{1 - z^{-1}}{T}$$

For the trapezoidal approximation,

$$Y(z) = \frac{T}{2} [X(z) + z^{-1} X(z)] + z^{-1} Y(z)$$

$$\text{or } H(z) = \frac{2}{T} \frac{1 - z^{-1}}{1 + z^{-1}}$$

$$\therefore s = \frac{2}{T} \frac{1 - z^{-1}}{1 + z^{-1}}$$

(b) For the first mapping, $z = \frac{1}{1 - sT}$

$$s = 0 \text{ gives } z = 1$$

$$\begin{aligned} \text{For } s = j\omega, \quad z &= \frac{1}{1 - j\omega T} = \frac{1}{2} \left[1 + \frac{1 + j\omega T}{1 - j\omega T} \right] \\ &= \frac{1}{2} \left[1 + e^{j2 \tan^{-1} \omega T} \right] \end{aligned}$$

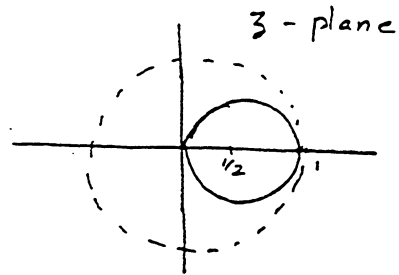
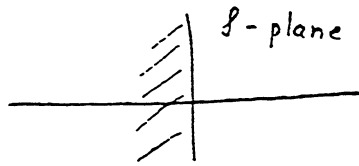
which is a circle with center $z = \frac{1}{2}$ and radius $\frac{1}{2}$

$$\text{For } s = \sigma + j\omega, \quad z = \frac{1}{2(1 - \sigma T)} \left[1 + e^{j2 \tan^{-1} \omega T} \right]$$

So that points along the line $\text{Re } s = \sigma$ map into a circle with radius $\frac{1}{2(1 - \sigma T)}$ and center $z = \frac{1}{2}$

Thus points in the left-half s -plane map into the interior of the circle with center $z = \frac{1}{2}$, radius $= \frac{1}{2}$ while points in right half-plane

map to the exterior



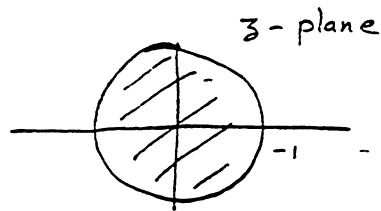
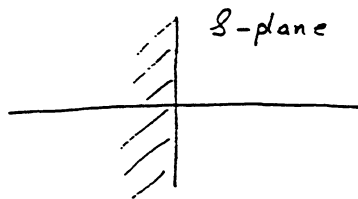
For $z = \frac{1 + \frac{T}{2}s}{1 - \frac{T}{2}s}$, when $s = j\omega$, we have

$$z = \frac{1 + j\omega \frac{T}{2}}{1 - j\omega \frac{T}{2}} = e^{2j \tan^{-1} \frac{\omega T}{2}}$$

So that the imaginary axis of the s -plane maps into the unit circle in the z -plane.

For $s = \sigma$, we have $z = \frac{1 + \sigma T/2}{1 - \sigma T/2}$

So that $|z| < 1$ if $\sigma < 0$ and $|z| > 1$ if $\sigma > 0$
 i.e. The left-half s -plane maps into the interior of the unit circle and the right-half plane to the exterior



$$9.1 (a) X(k) = e^{-j \frac{2\pi}{N} k n_0} \quad 0 \leq k \leq N-1$$

$$(b) X(k) = \sum_{n=0}^{N-1} (-1)^n e^{-j \frac{2\pi}{N} n k} = \sum_{n=0}^{N-1} (e^{j\pi} e^{-j \frac{2\pi}{N} k})^n$$

$$\text{For } N \text{ odd, } X(k) = \begin{cases} 2/(1 + e^{-j \frac{2\pi}{N} k}) & k \neq 0 \\ 1 & k = 0 \end{cases}$$

$$\text{For } N \text{ even, } X(k) = \begin{cases} N & k = N/2 \\ 0 & \text{otherwise} \end{cases}$$

(c) We can assume N is even. Then

$$X(k) = \sum_{n=0}^{N-1} e^{-j \frac{2\pi}{N} n k} = \sum_{m=0}^{\frac{N}{2}-1} e^{-j \frac{4\pi}{N} m k}$$

$$= \begin{cases} \frac{N}{2} & k=0, \frac{N}{2} \\ 0 & \text{otherwise} \end{cases}$$

$$9.2 \quad X(N-k) = \sum_{n=0}^{N-1} x(n) e^{-j \frac{2\pi}{N} n(N-k)}$$

$$= \left[\sum_{n=0}^{N-1} x(n) e^{-j \frac{2\pi}{N} n k} \right]^* = X^*(k)$$

$$9.3 (i) \quad Y_1(k) = \sum_{\substack{n=0 \\ n \text{ even}}}^{2N-1} x\left(\frac{n}{2}\right) e^{-j \frac{2\pi}{N} k n} \quad \text{Let } m = \frac{n}{2}$$

$$Y_1(k) = \sum_{m=0}^{N-1} x(m) e^{-j \frac{2\pi}{N} k m} = X(k)$$

Note that $Y_1(k)$ is a periodic sequence with period $2N$, while $X(k)$ is periodic with period N . That is,

$Y_1(k)$ consists of two complete periods of $X(k)$

$$(ii) \quad Y_2(k) = \sum_{n=0}^{N-1} x(N-n-1) e^{-j \frac{2\pi}{N} k n} = \sum_{m=0}^{N-1} x(m) e^{-j \frac{2\pi}{N} k (N-n-1)}$$

$$= e^{j \frac{2\pi}{N} k} X(-k)$$

$$(iii) \quad Y_3(k) = \sum_{n=0}^{\frac{N}{2}-1} x(2n) e^{-j \frac{2\pi}{N/2} n k} = \sum_{m=0}^{N-1} x(m) \left[\frac{1 + (-1)^m}{2} \right] e^{-j \frac{2\pi}{N/2} \frac{m}{2} k}$$

$$= \frac{1}{2} \sum_{m=0}^{N-1} x(m) e^{-j \frac{2\pi}{N} m k} + \frac{1}{2} \sum_{m=0}^{N-1} x(m) e^{-j \frac{2\pi}{N} (k + \frac{N}{2}) m}$$

$$= \frac{1}{2} X(k) + X(k + \frac{N}{2})$$

$$(iv) \quad Y_4(k) = \sum_{n=0}^{N-1} x(n) e^{-j \frac{2\pi}{2N} n k} = \sum_{n=0}^{N-1} x(n) e^{-j \frac{2\pi}{N} (\frac{k}{2}) n}$$

$$= \begin{cases} X(\frac{k}{2}) & \text{for } k \text{ even} \\ 0 & \text{otherwise} \end{cases}$$

$$\begin{aligned}
9.4 \quad Y(k) &= \sum_{n=0}^{16} x(n)W_{16}^{nk} = \sum_{n=0}^7 x(n)W_{16}^{nk} + \sum_{n=8}^{15} x(n-8)W_{16}^{nk} \\
&= \sum_{n=0}^7 x(n)W_8^{n(\frac{k}{2})} + [\sum_{n=0}^7 x(n-8)W_8^{n(\frac{k}{2})}]e^{-j\pi k} \\
&= (1 + e^{-j\pi k})X(\frac{k}{2}) = \begin{cases} 2X(\frac{k}{2}) & k \text{ even} \\ 0 & k \text{ odd} \end{cases}
\end{aligned}$$

so that

$$\begin{aligned}
Y(0) &= 2, Y(2) = Y^*(14) = 2 + j4, Y(4) = Y^*(6) = Y(10) = Y^*(12) = 2 - j2 \\
Y(8) &= 2 \text{ with all other } Y(k) = 0, 0 \leq k \leq 15.
\end{aligned}$$

$$9.5 \text{ (a)} \quad X(k) = 1 - e^{-j\frac{\pi}{3}k} - e^{-j\frac{2\pi}{3}k} + e^{-j\pi k} - e^{-j\frac{4\pi}{3}k} + e^{-j\frac{5\pi}{3}k}$$

so that

$$X(0) = 0, X(1) = 1 + j\sqrt{3} = X^*(5), X(2) = 3 + j\sqrt{3} = X^*(4), X(3) = -2$$

Also

$$H(k) = 1 + 2e^{-j\frac{\pi}{3}k} + 3e^{-j\frac{2\pi}{3}k} + 3e^{-j\pi k} + 2e^{-j\frac{4\pi}{3}k} + e^{-j\frac{5\pi}{3}k}$$

which gives

$$H(0) = 12, H(1) = -3 - j\sqrt{3} = H^*(5), H(2) = H(3) = H(4) = 0$$

so that

$$Y(1) = -j4\sqrt{2} = Y^*(5), Y(0) = Y(2) = Y(3) = Y(4) = 0.$$

Hence

$$\begin{aligned}
y_p(n) &= \frac{1}{6} \sum_{k=0}^5 Y(k)e^{j\frac{\pi}{3}kn} \\
&= \{0, 2, 2, 0, -2, -2\}
\end{aligned}$$

9.5 (ii)
$$X(k) = 1 - 2e^{-j\frac{\pi}{2}k} - e^{-j\pi k} + e^{-j\frac{3\pi}{2}k}$$

so that

$$X(0) = -1, X(1) = 2 + j3 = X * (3), X(2) = 1$$

Since $H(k) = 1 + e^{-j\frac{3\pi}{2}k}$, we have

$$H(0) = 2, H(1) = 1 + j = H * (3), H(2) = 0$$

and

$$Y(0) = -2, Y(1) = -1 + j5 = Y * (3), Y(2) = 0$$

so that

$$\begin{aligned} y_p(n) &= \frac{1}{4} \sum_{k=0}^3 \hat{Y}(k) e^{j\frac{\pi}{2}kn} \\ &= \{-1, -3, 0, 2\} \end{aligned}$$

9.6 (a) Zero-pad both sequences to length 11. Then, we have

$X(0) = 0$	$H(0) = 12$
$X(1) = X * (10) = -0.7036 + j0.9345$	$H(1) = H * (10) = 1.2326 - j8.5729$
$X(2) = X * (9) = 1.2635 + j1.4979$	$H(2) = H * (9) = -2.4556 - j0.7210$
$X(3) = X * (8) = 1.0211 + j1.4026$	$H(3) = H * (8) = 0.1108 - j0.2426$
$X(4) = X * (7) = 4.0133 + j0.4167$	$H(4) = H * (7) = 0.2835 + j0.1822$
$X(5) = X * (6) = -0.0944 - j2.9141$	$H(5) = H * (6) = 0.2328 - j0.3793$

$$Y(k) = H(k)X(k) \text{ gives } Y(0) = 0, Y(1) = Y * (10) = 7.1438 + j7.1838$$

$$Y(2) = Y * (9) = -2.0227 - j4.5894, Y(3) = Y * (8) = 0.4534 - j0.0923,$$

$$Y(4) = Y * (7) = 1.062 + j0.8494, Y(5) = Y * (6) = -1.1365 - j0.9221$$

so that

$$\begin{aligned} y_l(n) &= \frac{1}{11} \sum_{k=0}^{10} Y(k) e^{j\frac{2\pi}{11}kn} \\ &= \{1, 1, 0, -1, -3, -2, -1, 1, 2, 1, 1\} \end{aligned}$$

9.6 (ii) Zero-pad both sequences to length 7. Then

$$X(0) = -1$$

$$H(0) = 2$$

$$X(1) = 0.9254 + j2.1047 = X^*(6) \quad H(1) = 0.099 - j4.3339 = H^*(6)$$

$$X(2) = 2.9695 + j2.2978 = X^*(5) \quad H(2) = 1.625 + j0.7818 = H^*(5)$$

$$X(3) = 1.9559 - j0.88990 = X^*(4) \quad H(3) = 0.7775 - j0.9749 = H^*(4)$$

$$Y(k) = H(k)X(k) \text{ gives } Y(0) = -2, Y(1) = 0.8216 + j0.61 = Y^*(6),$$

$$Y(2) = 3.0245 + j6.0521 = Y^*(5), Y(3) = 0.654 - j0.25981 = Y(4)$$

so that

$$y_l(n) = \frac{1}{7} \sum_{k=0}^6 Y(k) e^{j \frac{2\pi}{7} kn}$$

$$= \{1, -2, -1, 2, -2, -1, 1\}$$

9.7 $X(\Omega) = \sum_{n=0}^{\infty} x(n) e^{-j\Omega n}$ Let $n = 8m + p$

$$X(\Omega) = \sum_{p=0}^7 \sum_{m=0}^8 x(8m+p) e^{-j\Omega(8m+p)}$$

$$X\left(\frac{2\pi}{8}k\right) = \sum_{p=0}^7 \sum_{m=0}^8 \left(\frac{1}{3}\right)^{8m+p} e^{-j \frac{2\pi}{8} k (8m+p)}$$

$$= \sum_{p=0}^7 \left[\left(\sum_{m=0}^8 \left(\frac{1}{3}\right)^{8m} \right) \left(\frac{1}{3}\right)^p \right] e^{-j \frac{2\pi}{8} k p}$$

$$= \sum_{p=0}^7 \frac{1 - \left(\frac{1}{3}\right)^9}{1 - \frac{1}{3}} \left(\frac{1}{3}\right)^p e^{-j \frac{2\pi}{8} k p}$$

$$\text{Let } y(p) = \frac{1 - \left(\frac{1}{3}\right)^9}{1 - \frac{1}{3}} \left(\frac{1}{3}\right)^p$$

$$\text{Then } X\left(\frac{2\pi}{8}k\right) = \sum_{p=0}^7 y(p) e^{-j \frac{2\pi}{8} k p} = X(k)$$

$$k = 0, 1, \dots, 7$$

$$\begin{aligned}
 9.8 \quad \sum_{n=0}^{N-1} |x(n)|^2 &= \sum_{n=0}^{N-1} x^*(n) \left[\frac{1}{N} \sum_{k=0}^{N-1} x(k) W_N^{-kn} \right] \\
 &= \frac{1}{N} \sum_{k=0}^{N-1} \left[\sum_{n=0}^{N-1} x(n) W_N^{kn} \right]^* x(k) \\
 &= \frac{1}{N} \sum_{k=0}^{N-1} X(k) X^*(k)
 \end{aligned}$$

9.9 (i) If $M > N$, augment $x(n)$ by adding $M - N$ zeros and take the M -point DFT.

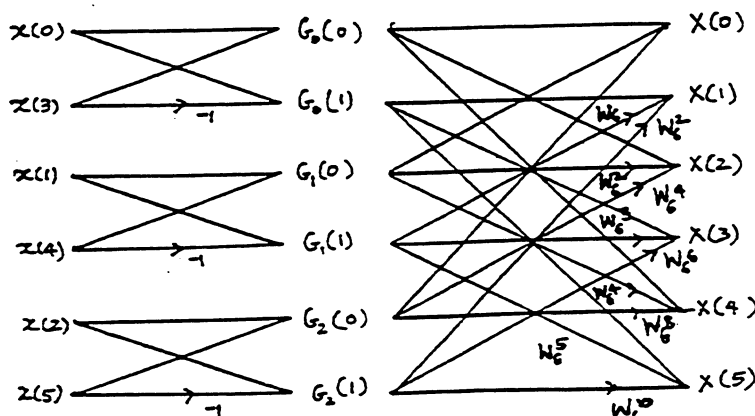
(ii) If $M < N$, let $M = NL$ where L is an integer and let $n = mM + p$. Then

$$\begin{aligned}
 X(e^{j\frac{2\pi}{M}k}) &= \sum_{n=0}^{ML} x(n) e^{-j\frac{2\pi}{M}kn} = \sum_{p=0}^{M-1} \sum_{m=0}^{L-1} x(mM + p) e^{-j\frac{2\pi}{M}(mM+p)k} \\
 &= \sum_{p=0}^{M-1} \left[\sum_{m=0}^{L-1} x(mM + p) \right] e^{-j\frac{2\pi}{M}pk} \\
 &= M\text{-point DFT of } \sum_{m=0}^{L-1} x(mM + p)
 \end{aligned}$$

9.10 128 samples in the range $[\frac{7\pi}{16}, \frac{15\pi}{16}]$ correspond to 512 points in the range $[0, 2\pi)$. Hence do a 512-point DFT. Then $k = 112$ corresponds to $\Omega = 7\pi/16$ rads and $k = 239$ corresponds to $\Omega = (239 \times 2\pi)/512 = [15\pi/16 - (2\pi)/512]$ rads. Use the procedure in Problem 9.9 to determine the 512-point DFT of $x(n)$ for the cases in (i) and (ii).

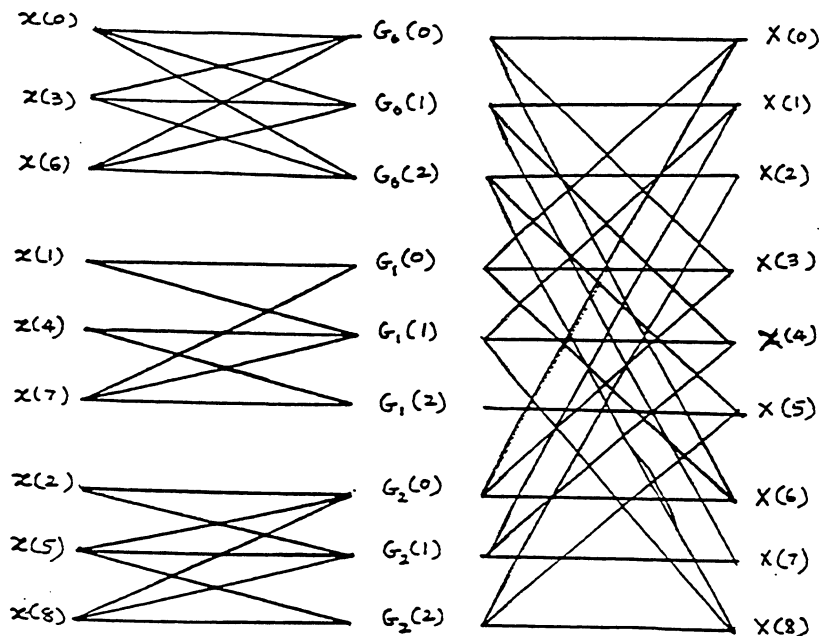
9.11 Use the results of Problem 9.9.

$$\begin{aligned}
 9.12 \quad X(k) &= \sum_{p=0}^5 x(n) W_6^{pk} \quad \text{Let } n = 3m + p \\
 X(k) &= \sum_{p=0}^5 \left[\sum_{m=0}^1 x(3m+p) W_2^{mk} \right] W_6^{pk} = \sum_{p=0}^5 G_p(k) W_6^{pk}
 \end{aligned}$$



$$9.13 \quad X(k) = \sum_{n=0}^8 x(n) W_9^{nk}$$

$$= \sum_{p=0}^2 \left[\sum_{m=0}^2 x(3m+p) W_3^{mk} \right] = \sum_{p=0}^2 G_p^k W_9^{pk}$$



$$9.14 \quad \Delta f = \frac{1}{T_0} < 0.25, \quad T_0 > 4 \text{ Sec}$$

$$9.15 \quad T_0 = NT = 4096 \times \frac{1}{40,000}$$

$$\Delta f = \frac{1}{T_0} = \frac{40,000}{4096} = 9.765 \text{ Hz}$$

$$9.16(a) \quad T_0 = 24, \quad \Delta f = \frac{1}{T_0} = \frac{1}{24} \text{ Hz}$$

$$(b) \quad N = \frac{T_0}{T} = f T_0 = \frac{128}{3} \times 24 = 1024$$

$$\therefore \Delta \Omega = \frac{2\pi}{N} = \frac{2\pi}{1024} \text{ radians}$$

(c) For no aliasing, analog signal must be band limited to $2\frac{1}{3} \text{ Hz}$.

$$\begin{aligned}
 9.17(a) \quad x(n) &= \frac{1}{16} \left[2 + (4-j4) e^{j \frac{2\pi}{16} 3n} - 2 e^{j \frac{2\pi}{16} 5n} \right. \\
 &\quad \left. - \frac{1}{2} e^{j \frac{2\pi}{16} 8n} - 2 e^{j \frac{2\pi}{16} 11n} + (4+j4) e^{j \frac{2\pi}{16} 13n} \right] \\
 &= \frac{1}{16} \left[2 + 4\sqrt{2} \cos\left(\frac{2\pi}{16} 3n - 45^\circ\right) - 2 \cos\left(\frac{2\pi}{16} 5n\right) \right. \\
 &\quad \left. + \left(-\frac{1}{2}\right)^n \right]
 \end{aligned}$$

$$(b) \quad \Delta\Omega = \frac{2\pi}{16} = \frac{\pi}{8} \text{ rads.}$$

$$(c) \quad T_0 = 4 \text{ secs}$$

$$(d) \quad \text{Highest frequency that does not cause aliasing} = 2 \text{ Hz.}$$

$$(e) \quad \Delta f = \frac{\Delta\Omega}{2\pi T} = \frac{1}{2} \text{ Hz}; \quad T_0 = \frac{1}{\Delta f} = 2 \text{ sec.}$$

$$9.18(a) \quad \text{If } h(n) \text{ is real, } H^*(k) = H(-k)$$

$$\begin{aligned}
 \therefore H_R(k) &= \frac{1}{2} H(k) + \frac{1}{2} H^*(k) \\
 &= \frac{1}{2} H(k) + \frac{1}{2} H(-k) = H_e(k) \\
 &= \frac{1}{2} H(k) + \frac{1}{2} H(N-k)
 \end{aligned}$$

$$\begin{aligned}
 j H_I(k) &= \frac{1}{2} H(k) - \frac{1}{2} H^*(k) \\
 &= \frac{1}{2} H(k) - \frac{1}{2} H(-k) = H_o(k) \\
 &= \frac{1}{2} H(k) - \frac{1}{2} H(N-k)
 \end{aligned}$$

$$(b) \quad \text{If } h(n) \text{ is purely imaginary, } H^*(k) = -H(-k)$$

$$\therefore H_R(k) = H_o(k) = \frac{1}{2} H(k) - \frac{1}{2} H(N-k)$$

$$j H_I(k) = H_e(k) = \frac{1}{2} H(k) + \frac{1}{2} H(N-k)$$

(c) Let $x(n) = f(n) + jg(n) = f(n) + h(n)$

Then $X(k) = X_R(k) + j X_I(k)$

$$= X_{Re}(k) + X_{Ro}(k) + j [X_{Ie}(k) + X_{Io}(k)]$$

Also $X(k) = F(k) + H(k)$

$$= F_R(k) + j F_I(k) + H_R(k) + j H_I(k)$$

Since $f(n)$ is real, $F_R(k)$ is even, $j F_I(k)$ is odd.

Since $h(n)$ is imaginary, $H_R(k)$ is odd, $j H_I(k)$ is even

$$\therefore F_R(k) = X_{Re}(k), \quad j F_I(k) = j X_{Io}(k)$$

$$H_R(k) = X_{Ro}(k), \quad j H_I(k) = j X_{Ie}(k)$$

That is $F(k) = \frac{X_R(k) + X_R(N-k)}{2} + j \frac{X_I(k) - X_I(N-k)}{2}$

$$H(k) = j G(k) = \frac{X_R(k) - X_R(N-k)}{2} + j \frac{X_I(k) + X_I(N-k)}{2}$$

$$\text{or } G(k) = \frac{X_I(k) - X_I(N-k)}{2} - j \frac{X_R(k) - X_R(N-k)}{2}$$

9.19 The Fourier transform of signal $x_a(t)$ is given by

9.20

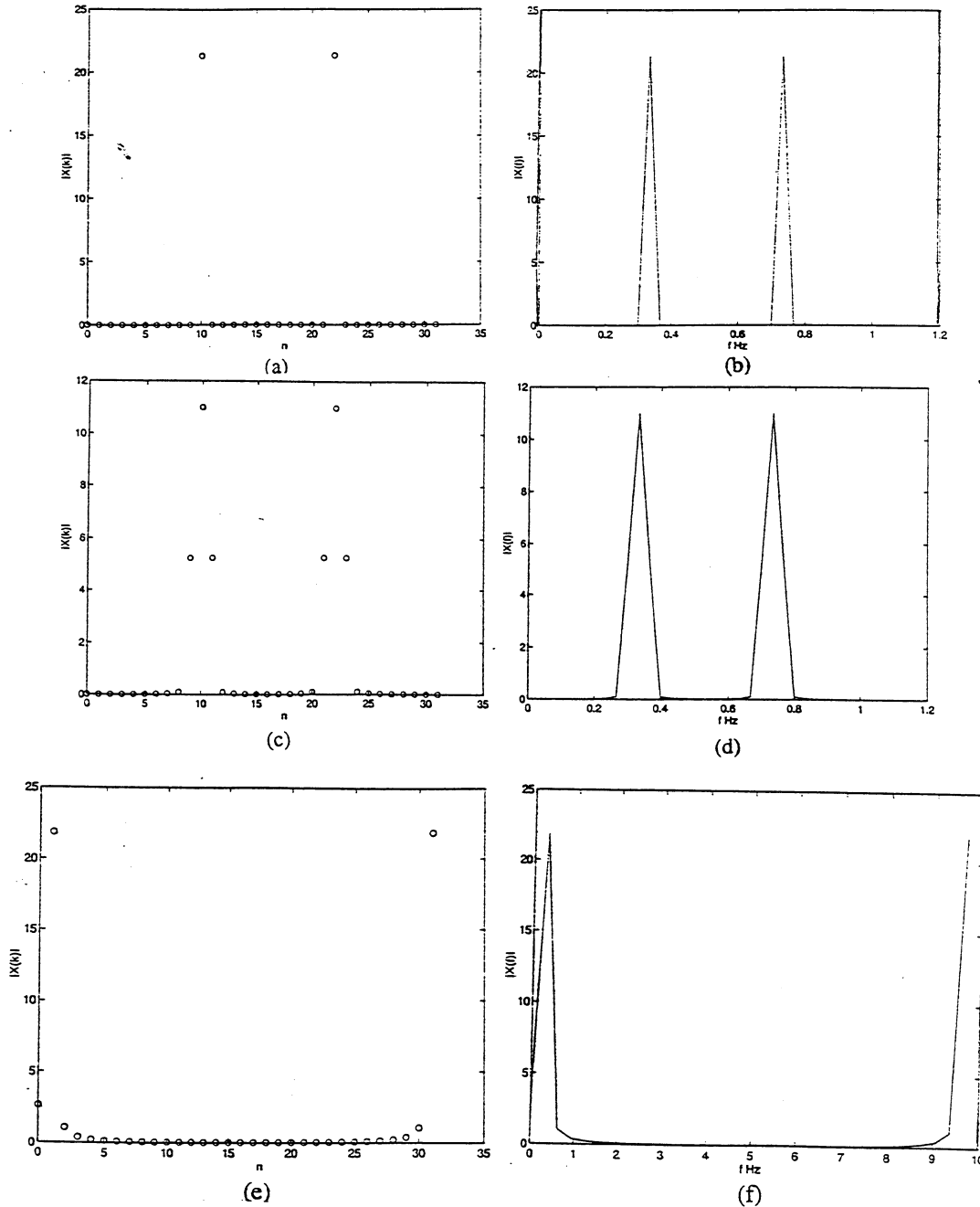
$$X_a(f) = 2\left[\delta\left(f - \frac{1}{3}\right) + \delta\left(f + \frac{1}{3}\right)\right]$$

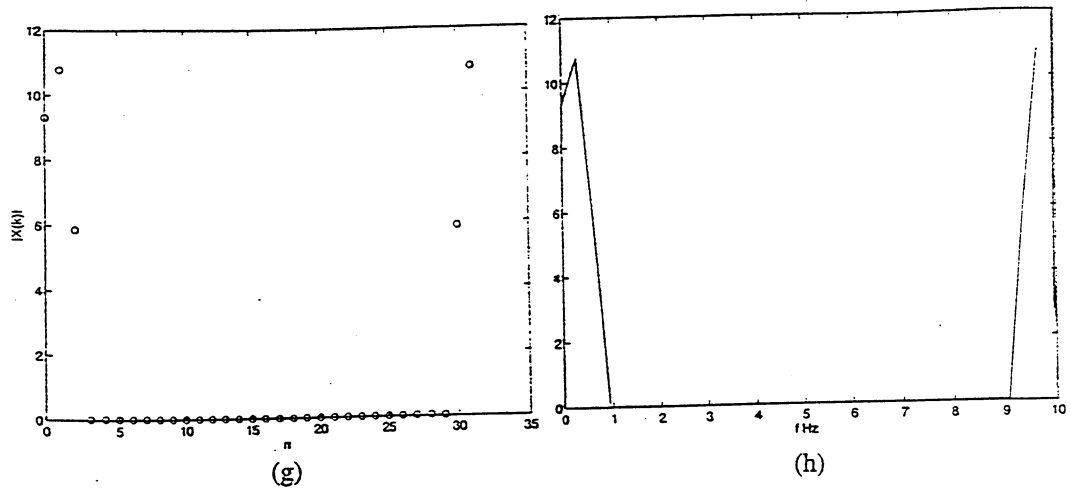
and consists of two δ -functions at $f = \pm 1/3$ Hz.

Figures (a)-(d) below show the results obtained using the DFT on a 32-point data sequence obtained by sampling $x_a(t)$ at intervals of $T = 15/16$ s, corresponding to a sampling frequency of $f_s = 16/15$ Hz. At this value of T , the duration of the analog signal used in computing the spectrum is 30 s. Figures (a) and (c) show plots of $|X(k)|$ versus k for the rectangular and Hamming windows respectively, while Figures (b) and (d) show plots of $|X(f)|$ versus f for $0 \leq f < f_s$ for the rectangular and Hamming windows.

Figures (e)-(h) below show the corresponding results obtained using the DFT when $x_a(t)$ is sampled at intervals of $T = 0.1$ s, corresponding to a sampling frequency of $f_s = 10$ Hz. The duration of the analog signal is 3.2 s or roughly 1/10th the duration in the previous case. Figures (e) and (g) show plots of $|X(k)|$ versus k for the rectangular and Hamming windows respectively, while Figures (f) and (h) show plots of $|X(f)|$ versus f for $0 \leq f < f_s$ for the rectangular and Hamming windows.

All the figures show two peaks, at frequencies of $1/3$ Hz and $f_s - 1/3$ Hz.





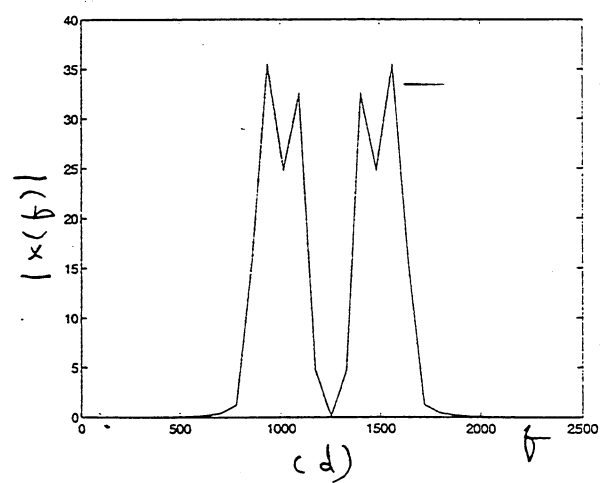
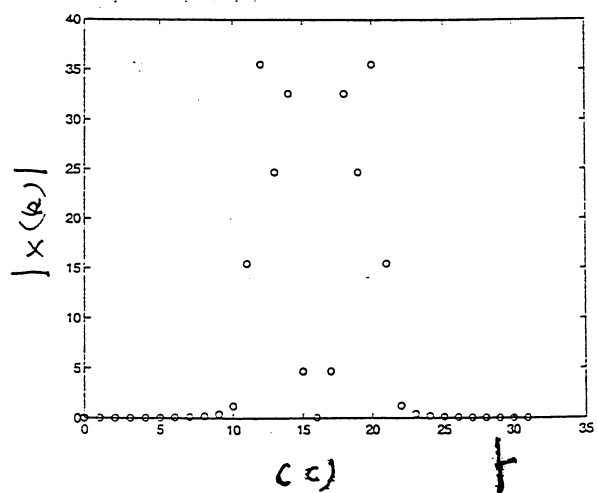
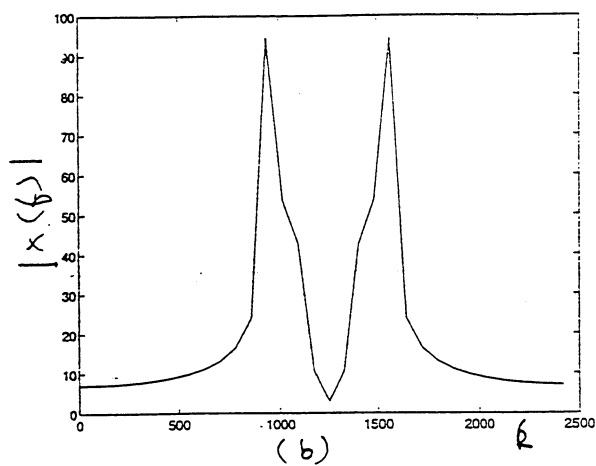
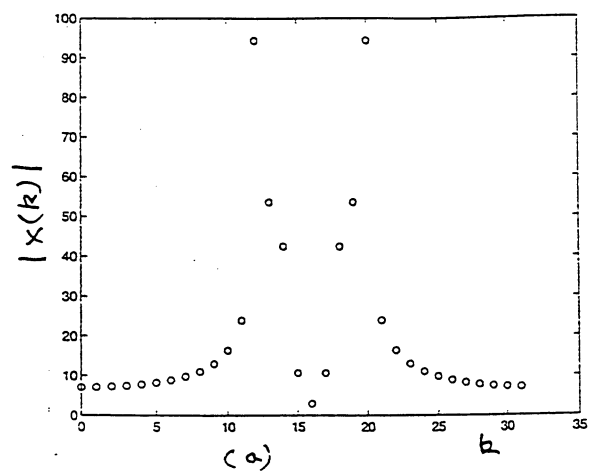
9.21

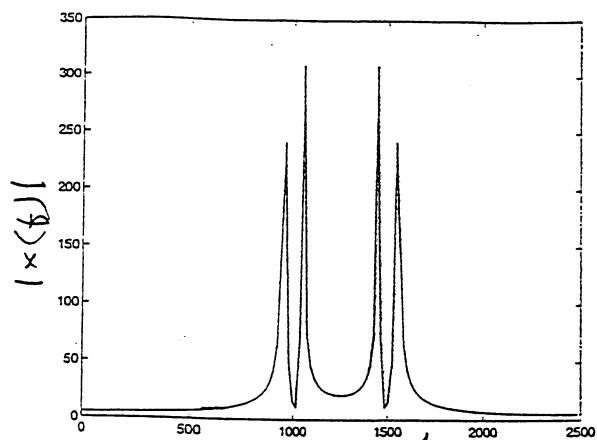
9.22

Since $x_a(t) = 5\cos(1900\pi t) + 5\cos(2100\pi t)$, $\Delta f = 100$ Hz. Thus $T_0 > 0.1$ sec. The sampling rate of $T = 0.4$ msec corresponds to a sampling frequency of $f_s = 2500$ Hz and satisfies the Nyquist criterion. With this value of T , the minimum number of samples required to resolve the two frequencies is $N = T_0/T = 25$. To use a radix-2 FFT algorithm, we therefore need at least 32 samples.

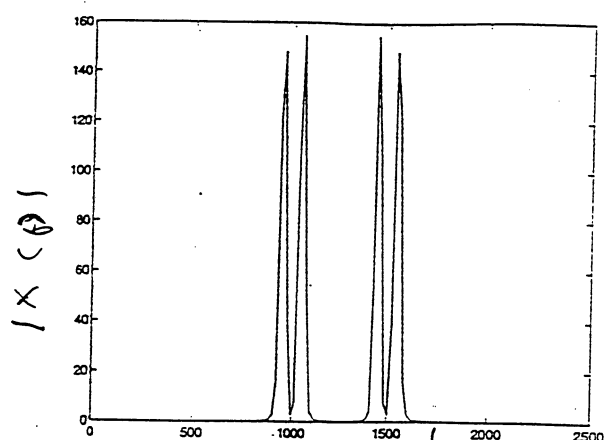
Figures (a) and (b) show the spectra obtained using a data length of 32 using a rectangular window. Figure (a) plots $|X(k)|$ vs k while (b) shows a plot of $|X_s(f)|$ vs f in the range $0 \leq f < f_s$. Figures (c) and (d) show the corresponding results using a Hamming window. As can be seen from the plots, for a data length of $N = 32$, we are unable to resolve the two frequencies using a rectangular window.

Figures (e) through (j) show corresponding results for data lengths of $N = 128, 256$ and 512 respectively. The figures on the left correspond to the rectangular window and those on the right to the Hamming window. For these data lengths, we can resolve the two frequencies with both windows. The peaks in the spectrum are narrower for the Hamming window. As the data length increases, the peaks become sharper.

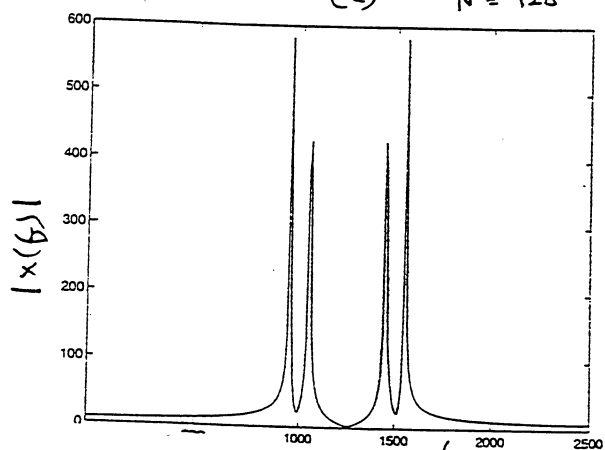




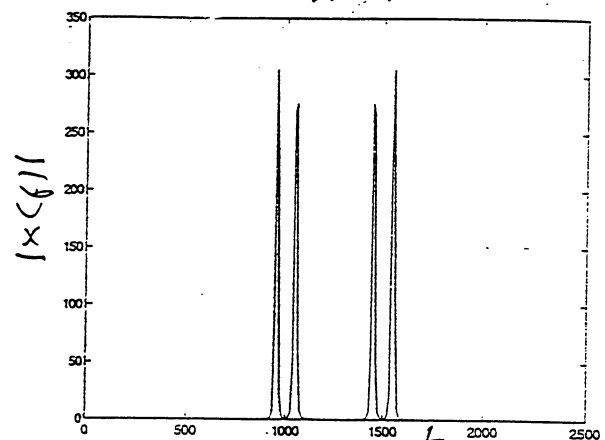
(c) $N = 128$



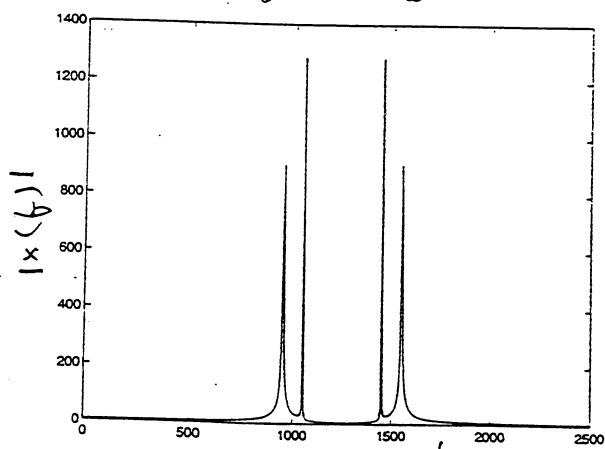
(d) $N = 128$



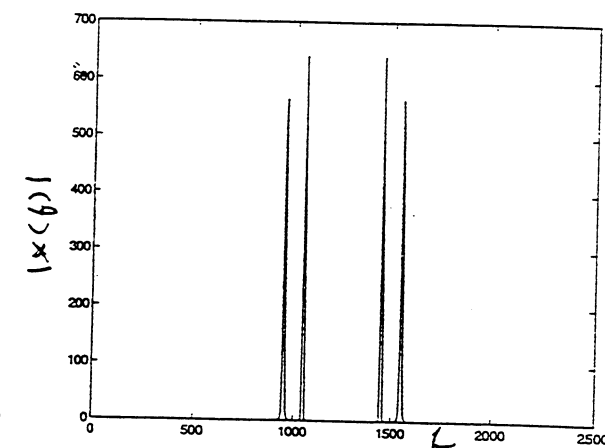
(e) $N = 256$



(f) $N = 256$



(g) $N = 512$



(h) $N = 512$

CHAPTER 10

- 10.1 Set $20\log_{10}(1-\delta_1) = -1.5$ and $20\log_{10}\delta_2 = -15$ to get $\delta_1 = 0.1586$ and $\delta_2 = 0.1788$.

Equation (10.3.15) gives $N = 1.5535$. We choose $N = 3$. If we use Equation (10.3.13), we get $\omega_c = 1412.5$ rad/s. The unnormalized filter is

$$H(s) = \frac{(1412.5)^2}{s^2 + \sqrt{2}(1412.5)s + (1412.5)^2}$$

- 10.2 With $f_p = 3$ kHz and $N = 2$, we solve for ω_c as

$$\omega_c = \left[\frac{1-\delta_1}{\sqrt{2}\delta_1 - \delta_1^2} \right]^{\frac{1}{N}} (2\pi f_p) = 26622 \text{ rad/sec}$$

and

$$f_s = f_c \left[\frac{1}{\delta_2^2} - 1 \right]^{\frac{1}{2N}} = 62631 \text{ rad/sec}$$

The filter transfer function is

$$H(s) = \frac{(26622)^2}{s^2 + \sqrt{2}(26622)s + (26622)^2}$$

- 10.3 We first design a normalized low pass filter with

$$20\log_{10}(1-\delta_1) = -1, \quad 20\log_{10}\delta_2 = -10$$

so that $\delta_1 = 0.108479$ and $\delta_2 = 0.31623$. Use of Eq. (10.13.15) gives

$N = 1.012$. Set $N = 2$ to get

$$H(j) = \frac{1}{j^2 + 1.41 + 2j + 1}$$

The required band pass filter is obtained by replacing

$$\text{by } \frac{\omega_0}{13W} \left(\frac{j^\#}{\omega_0} + \frac{\omega_0}{j^\#} \right) = \sqrt{6} \left(\frac{j^{\#2} + 6 \times 10^4}{\sqrt{6} 100 j^\#} \right)$$

$$\therefore H(j^\#) = \frac{1}{\left(\frac{j^{\#2} + 6 \times 10^4}{100 j^\#} \right)^2 + 1.414^2 \left(\frac{j^{\#2} + 6 \times 10^4}{100 j^\#} \right) + 1}$$

i.e $H(j) = \frac{10^4 j^2}{j^4 + 141.42j^3 + 12.01 \times 10^4 j^2 + 8.485 \times 10^6 j + 36 \times 10^8}$
by replacing j^ω by j .

10.4 Normalize by setting $\omega_p = 1$ so that $\omega_s = \frac{10}{3}$

$$20 \log_{10} \frac{1}{\sqrt{1+\epsilon^2}} = -2 \text{ gives } \epsilon = .7648$$

Also $20 = 20 \log_{10}(.7648) + 6(N-1) + 20N \log_{10}(\frac{10}{3})$
gives $N = 1.7213$

Choose $N=2$. Also $\beta = \frac{1}{2} \sin^{-1} \frac{1}{.7648} = .5415$

$\therefore$ Poles of Chebyshev filter are at

$$s = \frac{-1500}{\sqrt{2}} \times .5684 \pm j \frac{1500}{\sqrt{2}} \times 1.1502$$

$$= -602.88 \pm j 1219.97$$

$$\therefore H(s) = \frac{1}{(s + 602.88)^2 + (1219.97)^2}$$

10.5 The normalized Butterworth filter of order 3 is

$$H_B(s) = \frac{1}{(s^2 + s + 1)(s+1)} = \frac{1}{[(s + \frac{1}{2})^2 + (\frac{\sqrt{3}}{2})^2](s+1)}$$

To find the normalized Chebyshev filter, set

$$|H(1)| = \frac{1}{\sqrt{1+\epsilon^2}} = \frac{1}{\sqrt{2}} \text{ so that } -\epsilon = 1$$

so that $\beta = \frac{1}{3} \sinh^{-1}(1) = .2938$

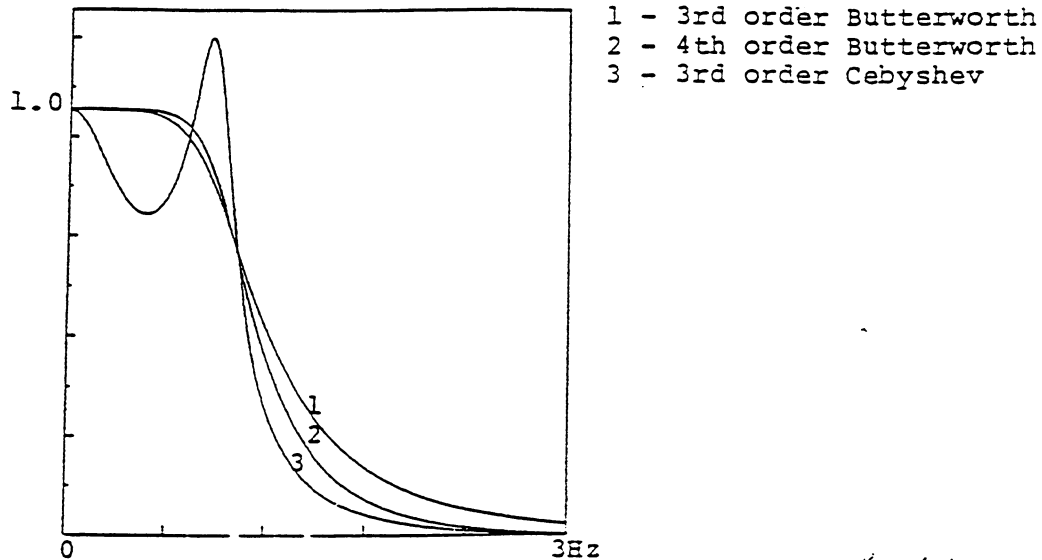
Complex roots of the Chebyshev filter are given by

$$s = -\frac{1}{2} \sinh(.2938) \pm j \frac{\sqrt{3}}{2} \cosh(.2938)$$

$$= -.149 \pm j .9036$$

Real root is $\lambda = -\sinh(0.2938) = -0.298$

$$\therefore H_C(s) = \frac{0.0723}{(s + 0.298)[(s + 0.149)^2 + (0.9036)^2]}$$



10.6 From the above figure, need a 5th order Butterworth filter.

10.7 From the specifications, $20 \log_{10} \frac{1}{\sqrt{1+\epsilon^2}} = -1.5$, so that $\epsilon = 0.4825$.

Equation (10.3.22) gives

$$20 \log_{10}(0.4125) + 6(N-1) + 20N \log_{10}(5) = 15$$

from which $N = 1.0703$. We set $N = 2$, so that

$$\frac{s}{\omega_p} = -\frac{1}{\sqrt{2}} \pm j \frac{1}{\sqrt{2}}$$

where $\omega_p = 2\pi 1000 = 6283.2$ rad/sec.

From Equation (10.3.26) $\beta = \frac{1}{2} \sinh^{-1} \frac{1}{0.4125} = 1.6187$, so that the poles of $H(s)$ are at

$$\begin{aligned} s &= -\frac{6283.2}{\sqrt{2}} \sinh(1.6187) \pm j \frac{6283.2}{\sqrt{2}} \cosh(1.6187) \\ &= -1.077 \times 10^5 \pm j 1.1651 \times 10^5 \end{aligned}$$

We therefore have

$$H(s) = \frac{1}{(s + 1.077 \times 10^5)^2 + (1.1651 \times 10^5)^2}$$

10.8

$$H(s) = \frac{(1412.5)^2}{s^2 + \sqrt{2}(1412.5)s + (1412.5)^2}$$

From Table 10.5,

$$H(z) = \frac{ze^{-aT} \sin(\omega_0 T)}{z^2 - 2ze^{-aT} \cos(\omega_0 T) + e^{-2aT}}$$

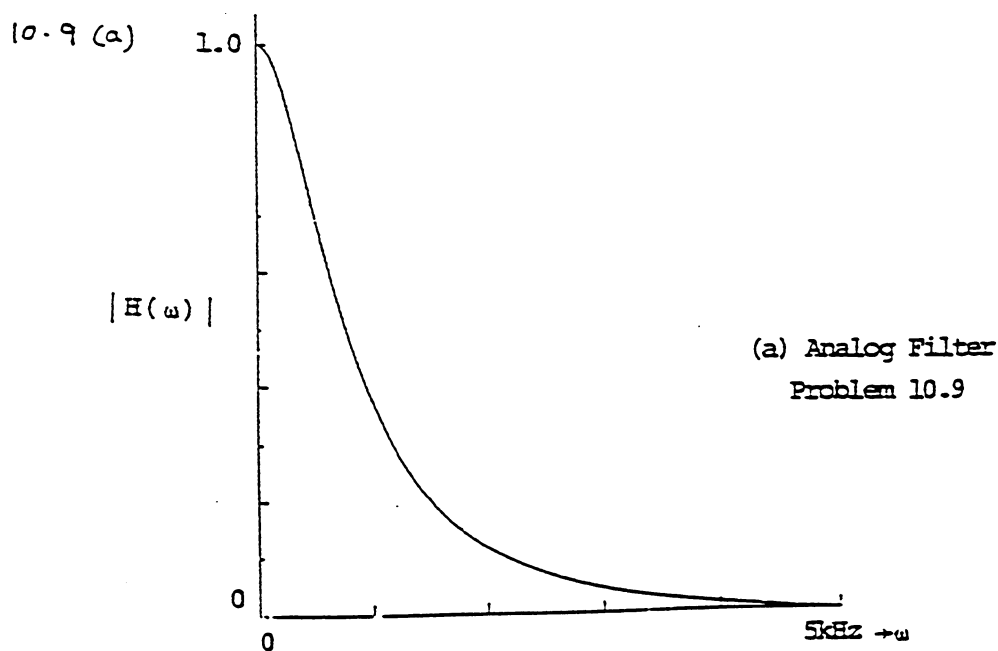
where $\omega_0 = a = \frac{\omega_c}{\sqrt{2}} = 998.78$.

With $T = \frac{1}{6000}$,

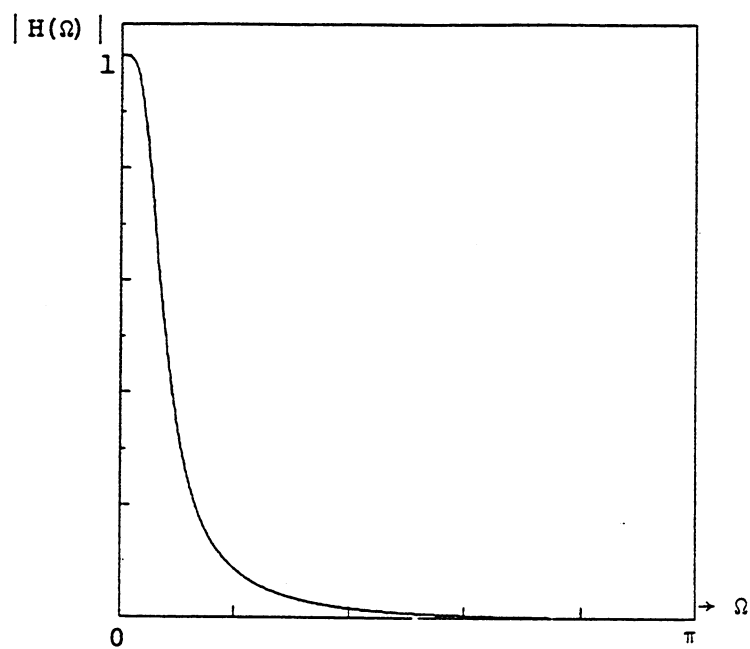
$$H(z) = \frac{.403z}{z^2 - 1.67z + .7168}$$

With $T = \frac{1}{10000}$,

$$H(z) = \frac{.0902z}{z^2 - 1.801z + .8189}$$

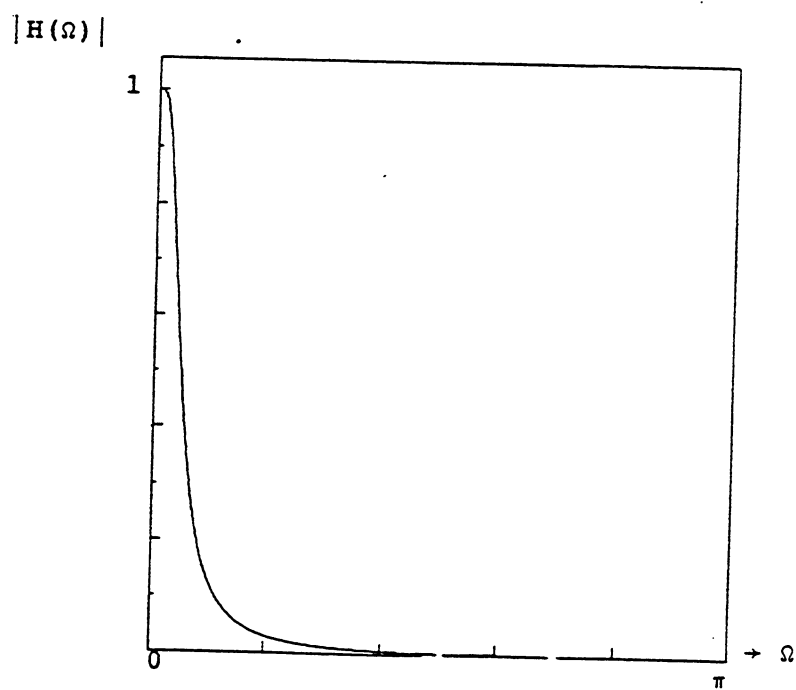


(b)



(b) Digital Filter with $f_s = 6000$ Hz
Problem 10.9

(c)



(c) Digital filter with $f_s = 10$ kHz
Problem 10.9

$$10.10 \quad \omega_p^d = \frac{2}{1/3000} \tan \left(\frac{1000 \times \frac{1}{3000}}{2} \right) = 1009.36$$

$$\omega_s^d = \frac{2}{1/3000} \tan \left(\frac{3000 \times \frac{1}{3000}}{2} \right) = 3277.81$$

From Problem 10.1, $\delta_1 = 0.20567$, $\delta_2 = 0.31423$,

$N = 1.434$ Choose $N = 2$ and use Eq. (10.3.15)

to get $\omega_s = 1154.19$

$$\begin{aligned} \therefore H(p) &= \frac{k}{p^2 + \sqrt{2} (1154.19)p + (1154.19)^2} \\ &= \frac{k}{p^2 + 1632.27p + (1154.19)^2} \end{aligned}$$

$$\text{Set } p = \frac{2}{(1/3000)} \frac{1 - z^{-1}}{1 + z^{-1}} \text{ to get}$$

$$H(z) = \frac{0.02836 (z+1)^2}{z^2 - 1.471z + 0.5844}$$

10.11 From problem 10.3, the normalized low-pass filter is

$$H(j) = \frac{1}{j^2 + \sqrt{2}j + 1}$$

For $f_s = 300 \text{ Hz}$, the predistorted frequencies are

$$\omega_{c_1}^d = 600 \tan \left(\frac{200}{600} \right) = 207.75$$

$$\omega_{c_2}^d = 600 \tan \left(\frac{300}{600} \right) = 327.78$$

$$\text{so that } \omega_o^d = \sqrt{\omega_{c_1}^d \cdot \omega_{c_2}^d}$$

$$\text{B.W.} = \omega_{c_2}^d - \omega_{c_1}^d = 120.03$$

10.12 (a)

$$\begin{aligned}
 H(\Omega) &= \sum_{n=-\infty}^{\infty} h(n)e^{-j\Omega n} \\
 &= \sum_{n=-\infty}^{-1} h(n)e^{-j\Omega n} + h(0) + \sum_{n=1}^{\infty} h(n)e^{-j\Omega n} \\
 &= h(0) + \sum_{n=1}^{\infty} [h(n)e^{-j\Omega n} + h(-n)e^{j\Omega n}]
 \end{aligned}$$

If $h(n) = h(-n)$,

$$H(\Omega) = h(0) + \sum_{n=1}^{\infty} 2h(n)\cos(\Omega n)$$

and $H(\Omega)$ is purely real.

If $h(n) = -h(-n)$, $h(0) = 0$, so that

$$H(\Omega) = j \sum_{n=1}^{\infty} (-2)h(n)\sin(\Omega n)$$

and is purely imaginary.

(b) Let N be odd and let $g(n) = h(n + \frac{N-1}{2})$.

Then

$$G(\Omega) = e^{j(\frac{N-1}{2})\Omega} H(\Omega)$$

and

$$\text{Arg}G(\Omega) = (\frac{N-1}{2})\Omega + \text{Arg}H(\Omega)$$

If $h(n) = h(N-1-n)$, $g(n) = g(-n)$ and $\text{Arg}G(\Omega) = 0$, so that

$$\text{Arg}H(\Omega) = -(\frac{N-1}{2})\Omega$$

If $h(n) = -h(N-1-n)$, $g(n) = -g(-n)$ and $\text{Arg}G(\Omega) = -\frac{\pi}{2}$, so that

$$\text{Arg}H(\Omega) = -(\frac{N-1}{2})\Omega - \frac{\pi}{2}$$

10.13 (a)

$$\begin{aligned}
 h_d(n) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} j\Omega e^{j\Omega n} d\Omega \\
 &= -\frac{1}{\pi} \int_0^{\pi} \Omega \sin(\Omega n) d\Omega \\
 &= \begin{cases} 0 & n = 0 \\ \frac{(-1)^n}{n} & \text{otherwise} \end{cases}
 \end{aligned}$$

(b) With an 11-point response and a delay of 6 samples, we have

$$h(n) = \left[\frac{1}{5} \quad \frac{-1}{4} \quad \frac{1}{3} \quad \frac{-1}{2} \quad 1 \quad 0 \quad -1 \quad \frac{1}{2} \quad \frac{-1}{3} \quad \frac{1}{4} \quad \frac{-1}{5} \right]$$

so that

$$H(z) = \frac{1}{5}(1 - z^{-10}) - \frac{1}{4}(z^{-1} - z^{-9}) + \frac{1}{3}(z^{-2} - z^{-8}) - \frac{1}{2}(z^{-3} - z^{-7}) + (z^{-4} - z^{-6})$$

The 11-point Hanning window is

$$w_h(n) = [.067 \quad .25 \quad .5 \quad .75 \quad .933 \quad 1 \quad .933 \quad .75 \quad .5 \quad .25 \quad .067]$$

The corresponding impulse response of the differentiator is

$$h_h(n) = [.0134 \quad -.0625 \quad .1667 \quad -.375 \quad .933 \quad 0 \quad -.933 \quad .375 \quad -.1667 \quad .0625 \quad -.0134]$$

with transfer function

$$\begin{aligned}
 H_h(z) &= 0.0134(1 - z^{-10}) - .0625(z^{-1} - z^{-9}) + 0.1667(z^{-2} - z^{-8}) \\
 &\quad - 0.375(z^{-3} - z^{-7}) + 0.933(z^{-4} - z^{-6})
 \end{aligned}$$

10.14 (a) The ideal filter has impulse response $h(n) = \sin(\pi n/6)/\pi n$.

The 12-tap filter response with a delay of 6 is given by

$$h(n) = [0 \quad .0318 \quad .0689 \quad .1061 \quad .1378 \quad .1592 \quad 1 \quad .1592 \quad .1378 \quad .1061 \quad .0689 \quad .0318 \quad 0]$$

The impulse response of the filter with the Hanning window is

$$h_h(n) = [0 \quad .006 \quad .0268 \quad .0649 \quad .1019 \quad .1513 \quad 1 \quad .1513 \quad .1019 \quad .0649 \quad .0268 \quad .006 \quad 0]$$

Plots of $|H(\Omega)|$ and $|H_h(\Omega)|$ are as shown below. It is clear that a 12-tap filter does not provide a good approximation to the desired filter response.

